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Competitive Nonlinear Tariffs

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This paper generalizes the study of nonlinear tariffs, i.e., those depending nonlinearly on the quantity purchased, to the case of a symmetric oligopoly. Competitive equilibria and the corresponding tariffs are analyzed in a Cournot framework. Various equilibria are obtained, which depend both upon the number of competing suppliers and the choice of market parameters used to characterize the competitors' strategies. Buyers are classified by type, each selecting an optimal consumption level in response to the prevailing tariff. The phenomena of buyer self-selection found in monopoly nonlinear pricing and the scaling of equilibrium demand elasticity found in Cournot models both appear in the results. *Journal of Economic Literature* Classification Numbers: 022, 611.

1. INTRODUCTION

Nonlinear prices, i.e., those with marginal rates varying with the quantity purchased, are commonly found in many markets. Quantity discounts are available in purchasing most types of durable goods. Products or services with metered usage, such as rental cars, photocopiers, long distance telephone calls, and digital communications are typically offered with a variety of price plans, depending on the quantity consumed.

Recently nonlinear pricing has received considerable attention in the economic literature. Nonlinear tariffs charged by a monopoly supplier have been studied by M. Spence [10]; Stiglitz [11]; Goldman, Leland and Sibley [3]; Sobel [9]; and Mirman and Sibley [5]. The welfare ramifications

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of such pricing have been investigated by Spence [10] and Roberts [8]. These authors emphasize that barriers to entry on the supply side, complemented by barriers to resale on the demand side, enable sellers to increase profits (and social welfare) by charging a nonlinear tariff. These tariffs induce price discrimination based on the "self-selection" of optimal consumption levels by various buyer types.

In this paper we extend the theory of nonlinear tariffs to the case of a symmetric oligopoly consisting of n suppliers offering identical products or services and having perfect information about the distribution of buyers' preferences. Buyers are classified by type, based on their preferences for consumption quantity, while price discrimination by the suppliers is permitted only with respect to purchase size. Equilibrium models for the suppliers are analyzed in a Cournot-like framework, which assumes that each supplier takes certain attributes of his competitors' market share as fixed in optimizing his own tariff. Because the buyers' demand data is disaggregated by purchase quantity and buyers' type, the market shares descriptions are given by distributions (rather than a single number). The alternative characterizations of the competitors lead to various Cournot models. Their solutions result in various equilibrium pricing strategies, which vary from monopoly pricing to marginal cost pricing. The formulations and equilibrium strategies also depend explicitly on n, the number of competing firms. As might be expected, in the case $n = 1$ all reduce to monopoly pricing, while in the limiting case $n \to \infty$ all reduce to marginal cost pricing.

Prerequisites for nonlinear tariffs, which our models share with previous ones, are the absence of a resale market and the existence of a means by which the supplier can monitor quantity consumed. As is common in the nonlinear pricing literature, income effects are assumed negligible. Since all firms are identical, any equilibrium nonlinear tariff must be symmetric. The buyers' optimal consumption levels when faced with this tariff are consequently analogous to the monopoly case. As in the monopoly case, a competitive nonlinear tariff achieves a partial price discrimination with regard to the various buyer types even when direct discrimination by buyer type is infeasible due to regulatory restriction or lack of information.

The basic formulation is introduced in Section 2 and the alternative models are analyzed in Sections 3 and 4. An illustrative example is presented in Section 5, followed by concluding remarks in Section 6.

2. Formulation

We consider a static model of several competing firms selling a single homogeneous product that may be purchased in any quantity q by consumers. The firms are identical in all relevant respects, and their products are perfect substitutes. Each firm's variable costs are assumed to be separable and identical among buyers, i.e.,

$C(q) =$ the cost of supplying any buyer with purchase of size $q > 0$.

We assume that buyers have perfect information about the prices and substitutability of the products. Hence in equilibrium all firms will charge the same nonlinear tariff, defined as

$$R(q) = \text{total purchase price for } q \text{ units.}$$

Each buyer's entire consumption is to be purchased from a single firm. This behavior is optimal for the buyer if $R(q)$ is concave, a restriction which we shall impose in our analysis. Consequently, there is no viable resale market. Buyers are assumed to be essentially infinite in number, and income effects are assumed to be negligible. Buyers are classified by type, each type having a particular optimal consumption behavior. All sellers are assumed to have complete knowledge of the buyers' type distribution and preferences, but are permitted to price discriminatorily only on the basis of purchase quantity.

Buyers' Behavior

Our derivation of the buyers' optimality conditions closely parallels that of Goldman, et. al. [3]. A buyer's type will be identified by an index t; each type t representing "many" buyers. For each quantity q there is a maximum willingness to pay, or reservation price, associated with buyer type t.

$W(q; t) =$ maximum willingness to pay for the first q units of consumption by buyers of type t.

Alternatively, we can think in terms of the marginal willingness to pay for the qth unit,

$$w(q; t) = \partial W(q; t)/\partial q,$$

which corresponds to the inverse of the demand function for buyer type t. We assume that buyer types have a continuous cumulative distribution and hence, without loss of generality, we may further assume that the index t is uniformly distributed on the interval $[0, 1]$[1]. A natural way to think about the index t is as a fractile ranking of the population with respect to the

[1] Suppose, for instance that θ is any index of types that has cumulative distribution $G(\theta)$ over the population, then we could simply relabel buyer type θ with $t = G(\theta)$ to obtain a uniform distribution, substituting $G^{-1}(t)$ for θ in the willingness to pay function.

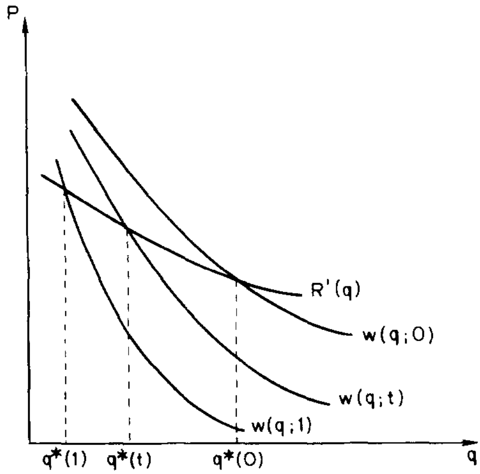

FIG. 1. Relationship of marginal tariff to demand curves.

particular attribute upon which the index is based. If this attribute is income, for example, then $t = 0.1$ corresponds to a buyer type that is marginally in the top 10% of the population with respect to income.

Each buyer type t has an optimal purchase size $q^*(t)$ that is obtained by maximizing his consumer surplus, i.e.;

$$q^*(t) = \arg\max_{q \geqslant 0} \{W(q; t) - R(q)\}. \tag{2.1}$$

However, he will become a subscriber[2] and make a purchase only if his optimal purchase quantity yields a nonnegative consumer surplus, i.e.,

$$CS(t) = W(q^*(t); t) - R(q^*(t)) \geqslant 0. \tag{2.2}$$

For analytical convenience, t will be treated as a continuous variable, although analogous constructions can be pursued in the discrete case. We will assume that $W(q; t)$ is a smooth function in both varibles and that $R(q)$ is continuously differentiable except at $q = 0$, where it may have a jump. We will also assume that the indices t are assigned in such a way that *smaller* t values correspond to buyer types that tend to prefer *larger* quantities; that is $\partial w/\partial t \leqslant 0$ for all q, t. At the same time marginal preference for quantity is also decreasing in q, i.e. $\partial w/\partial q \leqslant 0$ for all q, t.

[2] Throughout this paper we will use the term subscribers to distinguish buyers which meet condition (2.2) and hence are willing to pay the fixed charge $R(0^+)$. This term is usually associated with services to which a buyer may or may not subscribe. We will use it, however, in a more general sense for any type of purchases that may involve a fixed fee.

Thus, the first order conditions necessary for (2.1) may be written as:

$$\begin{aligned} w(q^*;t) &= R'(q) && \text{if } q^* > 0; \\ &\leqslant R'(0^+) && \text{if } q^* = 0^+. \end{aligned} \tag{2.3}$$

Following Goldman, et al [3] and Stiglitz [11] we assume that the marginal tariff $R'(q)$ intersects each inverse demand curve $w(q;t)$ just once from below (see Fig. 1). Then, based on Lemma 1 in Goldman [3], the solution to (2.3) yields a unique maximum $q^*(t)$ and it satisfies[3]

$$dq^*/dt \leqslant 0. \tag{2.4}$$

Since $q^*(t)$ is monotone we may define the inverse function,

$$t^*(q) = \max_t \{t \mid q^*(t) \geqslant q\}, \qquad q \geqslant 0, \tag{2.5}$$

corresponding to the index of the smallest potential buyer whose optimal purchase size is at least q.

In general, the set of subscribers may not include the entire unit interval. This is because, as stated in (2.2), a buyer of type t makes a purchase if and only if his consumer surplus $CS(t)$ is nonnegative at his optimal consumption level. The following results provide a useful characterization of the set of buyers meeting this requirement.

LEMMA 2.1. *The subscriber set is of the form* $[0, t_1]$, *where* $t_1 \leqslant 1$. *Furthermore, if this set is nonempty, then either* $t_1 = 1$ *or* $CS(t_1) = 0$.

Proof. Clearly the subscriber set is a subset of the interval $[0, t^*(0^+)]$, including all buyers having a positive $q^*(t)$. However, by the monotonicity condition (2.4), $t \geqslant t^*(0^+)$ implies $q^*(t) > 0$. Hence, equality holds in (2.3) which can be used to obtain

$$dCS(t)/dt = \int_0^{q^*} \{\partial w/\partial t\}\, dq < 0, \qquad \text{for} \quad t \in [0, t^*(0^+)). \tag{2.6}$$

Tnus, if $t_1 > 0$, then either $CS(t)$ is positive on the entire unit interval and $t_1 = 1$, or there is exactly one value t_1 such that $CS(t_1) = 0$, with $CS(t) > 0$ for $t < t_1$ and $CS(t) < 0$ for $t > t_1$. Q.E.D.

[3] When the willingness to pay function is strictly decreasing in t, i.e., $\partial w/\partial t < 0$, then (2.4) is a strict inequality. In this case t defines a customer ranking with regard to optimal purchase size that is *invariant* to the tariff $R(q)$. Since we assume that the buyers' type index t is uniformly distributed on the interval $[0, 1]$, the index t also defines a buyer type's percentile ranking with regard to purchase size. This interpretation is useful from an operational point of view since it allows customers to be classified directly on the basis of existing demand data.

Summary of Buyers' Behavior. Let $w(q; t)$ satisfy $\partial w/\partial t \leqslant 0$, $\partial w/\partial q \leqslant 0$ for all t, q, and let $R'(q)$ intersect $w(q, t)$ at most once from below. Then, there is an optimal purchase size function $q^*(t)$ satisfying $dq^*/dt \leqslant 0$, obtained from (2.3). The set of indices corresponding to subscribers is of the form $[0, t_1]$, where t_1 either equals one or satisfies the condition $W(q^*(t_1); t_1) = R(q^*(t_1))$. The index t_1 denotes the type of the marginal subscriber. However, since t is uniformly distributed on $[0, 1]$, t_1 may also be interpreted as the market share represented by the buyers who subscribe. The optimal purchase size of subscribers will satisfy $0 \leqslant q_1 \leqslant q^*(t) \leqslant q_0$, where the smallest purchase size is $q_1 = q^*(t_1)$ and the largest purchase size is $q_0 = q^*(0)$.

Sellers' Behavior

Each seller's goal is to maximize, within the competitive framework, his individual net revenue, subject to the optimal buyer behavior described above. Given the supply cost function $C(q)$ the sellers are to select an optimal tariff $R(q)$. We will assume that both these functions are positive, increasing and differentiable for $q > 0$. In a symmetric equilibrium, all sellers will have identical cost functions and tariffs. Hence, the total net revenue shared among the sellers is given by

$$NR[R] = \int_0^{t_1} [R(q^*(t)) - C(q^*(t))]\, dt, \tag{2.7}$$

where $q^*(t)$ is determined by (2.1).

An alternative expression, which we will refer to later, is obtained by introducing the change of variables $t = t^*(q)$, where the endpoints of q are the smallest and largest purchase sizes q_1 and q_0,

$$NR[R] = [R(q_0) - C(q_0)]\, t^*(q_0) - \int_{q_1}^{q_0} [R(q) - C(q)]\, dt^*(q). \tag{2.8}$$

This representation accounts for the possible discontinuities in $t^*(q)$. Thus the integral in (2.8) is a Riemann-Stieltjes integral on the interval $[q_1, q_0]$, while the first term accounts for a possible interval $[0, t^*(q_0)]$ of subscribers who all choose the largest purchase size.

For the monopolist seller, $R(q)$ is determined by maximizing $NR[R]$ with respect to $R(q)$, subject to the self-selection constraints given by (2.3) and Lemma 2.1. The solution to this problem, which has been treated by Spence [10] and Goldman, et al [3], results as a special case of our subsequent treatment of the oligopoly.

Several Alternative Cournot Models

In a symmetric Cournot equilibrium, each of the n competitors will ultimately obtain an equal share of the net revenue. This equilibrium is achieved through a competitive process in which each firm predicts the market share captured by his competitors, and optimizes his own tariff with respect to the unsatisfied demand, assuming static behavior by the competitors. An equilibrium is attained when each firm follows this process and makes the same correct static prediction about the others. Since predicting the customers' response to a nonlinear tariff employs disaggregated demand data, a firm's strategy and corresponding prediction about its competitors can also be defined in a disaggregated manner. The supply quantity which defines a firm's strategy in the standard Cournot model, is replaced here by a function that disaggregates quantity according to one of three possible market descriptors. These are: purchase size q, total purchase price R, and buyer's index t. The function itself may be measured in terms of number of purchases or total quantity purchased. This leads to six possible Cournot equilibrium models, each having its own equilibrium strategy. These models can be organized in the form of the three by two matrix in Table I.

Note that Models I and II are identical since, given the purchase size, determining either the total purchase quantity or number of buyers implies the other. In some industries the market characteristics may naturally suggest one of the above models, while in other situations several models may be plausible. The following is a brief description and organizational interpretation for each of the different models.

Model I (or II) corresponds to the situation in which firms define their marketing strategy in terms of how many orders of each size q each hopes to achieve. This is measured in terms of the "cumulative" market share of customers that order q units or more. For example, a cement company might think in terms of 5 cubic yard orders, 50 cubic yard orders, 200 cubic yard orders, and the fraction of its orders that would be for at least each of these amounts. Another natural example for this model is the photocopier leasing

TABLE I

Characterization of Models I–VI

	Number of purchases	Quantity purchased
Purchase size q	I	II
Purchase price R	III	IV
Buyers index t	V	VI

industry, in which firms often characterize their market share in terms of number of machine placements within given ranges of monthly copy volume. Typically, different types of machines are used for different "volume bands." Thus, in planning its marketing strategy a firm commits itself to the number of units of each machine type it intends to place and will adjust prices if necessary to meet this commitment. Each firm attempts to predict the number of units placed by its competitors in each volume band and takes this number as fixed when it optimizes its own strategy.

Model III is the purchase price analog of Model I. Here the firms define their strategies in terms of the number of orders in each price range, i.e. $1000 orders, $10,000 orders, etc. Model IV is like Model III except that the firms think in terms of total quantity sold at various dollar order sizes. That is, the cement company would define its strategy by the total number of cubic yards sold in orders of $10,000 or more, etc. The number of cubic yards that $10,000 buys will depend on the tariff, which is determined by the net revenue optimization.

Models V and VI focus on customer types, as ranked by their preference for quantity consumption. Model V implies that each firm defines its strategy in terms of the number of customers it wants in the top 10%, top 20%, top 50%, etc., ranked according to their consumption level. That is, each firm is setting the number of sales targeted to every buyer type. This model would be appropriate in some service markets in which customer loyalty is high, due to sunk costs on the part of the buyer or long-term contractual agreements. Thus, the main focus of the competition is the number of "captive" clients each firm has in each buyer's type category. Such a situation might occur in the electronic mail market, where the key aspect of a firm's market share is the number of its network subscribers, while tariffs are employed both to attract subscribers and to regulate their usage level.

Model VI is only slightly different from Model V in that the firm specifies the total quantity that is to be sold to each class of customers. Since customers are identified only by their type, this strategy does not define the quantity q that will be purchased by the various types. This will again depend upon the tariff, which is selected to optimize net revenues. This model would apply, for example, to a satellite communication industry where a firm's strategy might be defined by its allocation of channel capacity among various classes of subscribers (e.g. high, medium and low volume users).

Formulation of Models I, II and III

In Models I or II, the policy of firm i will be defined by the function

$$T_i(q) = \text{fraction of buyers purchasing } q \text{ units or more}, \quad q_1 \leqslant q \leqslant q_0.$$

Following a Cournot model, firm i will predict the total such sales of its competitors,

$$Y_i(q) = \sum_{j \neq i} T_j(q). \tag{2.9}$$

Firm i will assume that this cumulative market share captured by its competitors is fixed, and will attempt to maximize its own net revenue by selecting the tariff function $R(q)$. In view of (2.8), firm i's net revenue, conditional on $Y_i(q)$, is given by

$$NR_i[R_i; Y_i] = [R(q_0) - C(q_0)][t^*(q_0) - Y_i(q_0)] + \int_{q_0}^{q_1} [R(q) - C(q)]\, d[t^*(q) - Y_i(q)]. \tag{2.10}$$

In a symmetric Cournot equilibrium, all firms make the correct prediction about their competitors, leading to the symmetry condition

$$Y_i(q) = (n-1)\, T_i(q) = (n - 1/n)\, t^*(q) \qquad \text{for} \quad i = 1, \ldots, n. \tag{2.11}$$

This model will be solved and analyzed in Section 3. An interesting feature of the above model is that it relies only on demand data that is aggregated by purchase size. This makes it possible to derive the optimal nonlinear tariff without referring to customer types. Hence, the assumption of non-intersecting demand curves of different customers, or equivalently the monotonicity of the reservation price $w(q; t)$ in t is not needed.

In Model III firm i's strategy is defined by

$T_i(R) =$ fraction of buyers paying total price R or higher.

The corresponding prediction of competitive behavior is therefore the total sales, $Y_i(R)$, by the competition in each price range. The net revenue of firm i and the equilibrium conditions are again given by (2.10) and (2.11), respectively, with the exception that now $Y_i(q)$ is replaced by $Y_i(R(q))$.

Formulation of Models IV, V and VI

In Model IV, firm i's strategy is specified by the total quantity $Q_i(R)$ to be sold at purchase price of R or higher. The corresponding prediction about the competition is the quantity, $X_i(R)$, sold at purchase price of R or more. This is related to the function $Y_i(q)$ in Model I by the condition

$$dY_i(q) = \{dX_i(R(q))\}/q, \qquad q_1 \leqslant q \leqslant q_0. \tag{2.12}$$

On the right side, the competitive share in terms of the total quantity share purchased in the amount q, divided by q, equals the competitive share on the left side in terms of *orders* placed. Substituting (2.12) into (2.10) yields the net revenue of firm i, $NR[R_i; X_i]$, conditional on $X_i(R)$, which is now the fixed competitive strategy against which firm i optimizes its tariff. The symmetry conditions are again given by (2.11).

In Model V, firm i's strategy is characterized in terms of the number of subscribers it wants to have among buyers in the top t fractile, i.e., those whose index is between 0 and t. The appropriate prediction is therefore the total fraction of customers $S_i(t)$ that will be captured by its competitors in each interval $[0, t]$. Given this prediction, firm i's share of subscribers among the top t fraction of buyers will amount to $t - S_i(t)$ and its net revenue is given by

$$NR_i[R_i; S_i] = \int_0^{t_1} [R(q^*(t)) - C(q^*(t))]\, d[t - S_i(t)], \qquad (2.13)$$

where $q^*(t)$ and t_1 are selected optimally by the buyers as before. Firm i's response to his competitors' predicted behavior is to choose the $R(q)$ that maximizes (2.13), holding $S_i(t)$ fixed. A symmetric Cournot equilibrium will result when all firms make the same correct prediction about each other, which implies the symmetry condition

$$S_i(t) = (n - 1/n)\, t, \qquad i = 1, 2, \ldots, n. \qquad (2.14)$$

Model VI can be regarded as a variant of Model V. Rather than aiming at the number of customers of each type, the firm's strategy is now defined in terms of the *total* units $Q_i(t)$ sold to the top t fraction of customers. The corresponding prediction about the competition is the total quantity $X_i(t)$ sold to the top t cusomers. If the equilibrium tariff were $R(q)$, then the total demand of the top t customers would be $\int_0^t q^*(\tau)\, d\tau$. However, firm i predicts that $X_i(t)$ of that demand will be taken by the competition. Thus, its quantity sold to the top t customers is

$$Q_i(t) = \int_0^t q^*(\tau)\, d[\tau - X_i(\tau)]. \qquad (2.15)$$

Consequently, the number of orders from firm i by customers with exactly index t must satisfy a differential condition analogous to (2.12),

$$dS_i(t) = [1/q^*(t)]\, dX_i(t). \qquad (2.16)$$

Substituting for $dS_i(t)$ in (2.13) in terms of (2.16) yields firm i's net revenue $NR_i[R_i; X_i]$, when $X_i(t)$ is held fixed. The symmetry condition (2.14) still holds.

Models I and VI will be solved in detail in the next section. In Model V, firm i's optimization problem, given its prediction about competitors' market share, reduces to the monopolist's pricing problem. This occurs because conceding a fixed set of customers to its competitors, as the Cournot assumption implies here, is equivalent to believing that the remaining customers are "captive." Consequently, firm i's optimal strategy is to act as a monopolist toward its "guaranteed" share of the market. Model II can also be viewed as solved from reinterpreting Model I. Models III and IV have proved less tractible. Although calculus of variations techniques can be used to obtain necessary conditions for optimality, they appear complicated and uninterpretable. Therefore, we have omitted the analysis of these models from the paper.

3. Analysis of Model I

We now turn to the analysis of Model I. We will obtain a differential equation that the symmetric equilibrium tariff $R(q)$ must satisfy, and the corresponding boundary conditions, which are determined by optimizing the endpoints of the interval $[q_1, q_0]$. In this model, a firm's optimal response to the predicted composite $Y_i(q)$ of purchase size distribution from its competitors is to choose the tariff R that maximizes its net revenue given by (2.10). Since all firms are identical, we suppress the index i of the firm. Integrating (2.10) by parts and substituting in the buyers' self-selection conditions given by (2.3) and Lemma 2.1 yields

$$NR[R; Y] = [W(q_1; t_1) - C(q_1)][t_1 - Y(q_1)] + \int_{q_1}^{q_0} [t^*(\theta) - Y(\theta)][w(\theta; t^*(\theta)) - c(\theta)]\, d\theta, \tag{3.1}$$

where $c(q) = dC(q)/dq$ is the marginal cost. For $q_1 < q < q_0$, (3.1) can be maximized with respect to $t^*(q)$ by pointwise maximization of the integrand. The Euler first order necessary conditions for an interior local maximum are:

$$(t - Y)\, \partial w(q; t)/\partial t + w(q; t) - c(q) = 0, \qquad q_1 < q < q_0. \tag{3.2}$$

In a symmetric equilibrium, however, one expects that each of the n competitors will get $1/n$ of the total $t^*(q)$ for each q. Consequently

$$Y(q) = (1 - 1/n)\, t^*(q). \tag{3.3}$$

This can be substituted in (3.2) to yield an implicit equation defining $t^*(q)$ or, alternatively, $q^*(t)$,

$$(t/n)\, \partial w(q; t)/\partial t + w(q; t) - c(q) = 0. \tag{3.4}$$

The optimal tariff is determined by substituting $t^*(q)$ into (2.3) to obtain

$$R'(q) = w(q, t^*(q)), \tag{3.5}$$

and integrating

$$R(q) = \int_{q_1}^{q} w(\theta; t^*(\theta))\, d\theta + W(q_1; t^*(q_1)) \quad \text{for } q_1 \leqslant q \leqslant q_0. \tag{3.6}$$

Notice, since $\partial w/\partial t \leqslant 0$, that the solution to (3.4) always yields a nonnegative value for the integrand in (3.1). Furthermore, for any number of competitors, the largest purchase size $q_0 = q^*(0)$ must satisfy the equation $w(q_0, 0) = c(q_0)$ (unless $\partial w(q; t)/\partial t$ is unbounded at $t = 0$). By (3.5) it then follows that the last unit in the largest purchase is always priced at cost.

The boundary conditions involving t_1 are obtained from Lemma 2.1. Since $R(q)$ in (3.6) is defined parametrically on t_1, we complete the solution by optimizing t_1, the index of the smallest purchaser. The resulting first order necessary conditions for maximizing (3.1) with respect to t_1 are

$$[W(q_1; t_1) - C(q_1)] + (t_1 - Y(q_1))\, \partial W(q_1; t_1)/\partial t_1 = 0. \tag{3.7}$$

In a symmetric Cournot equilibrium $Y(q_1) = (1 - 1/n)\, t_1$, reducing (3.7) to:

$$[W(q_1; t_1) - C(q_1)] + (t_1/n)\, \partial W(q_1; t_1)/\partial t_1 = 0. \tag{3.8}$$

However, q_1 and t_1 must also satisfy (3.4) so they are completely determined by (3.4) and (3.8), For $n = 1$, these equations yield the monopoly solution that would have been obtained by maximizing (2.8).

Discussion

Equation (3.4) can be interpreted as a family of elasticity conditions holding for each q value. Let $p(q) = R'(q)$ be the marginal price and $N(p, q)$ be the demand for orders of quantity q or more, given the fixed marginal price p. Then the solution $t^*(q)$ to (3.5) must satisfy

$$w(q, t^*(q)) = p(q), \qquad t^*(q) = N(p(q), q). \tag{3.9}$$

Thus, if we define the elasticity

$$e_N(q) = -(\partial N/\partial p)(p/N)|_{p = p(q)}, \tag{3.10}$$

(3.4) may be written as

$$p(q)[1 - 1/(n e_N(q))] - c(q) = 0. \tag{3.11}$$

This is analogous to the classical Cournot elasticity relationship, in which the elasticity term is multiplied by n, the number of suppliers in the oligopoly. Here we have a family of such relationships, one for each q, which must hold simultaneously in equilibrium. As $n \to \infty$, Equation (3.11) reduces to $p(q) = c(q)$, yielding a marginal pricing rule which is optimal under perfect competition. Equation (3.8) can be used to verify that $R(q_1) = W(q_1; t_1) = C(q_1)$ in this case as well, so that $R(q) = C(q)$.

We make several additional remarks that are stated here in the context of Model I where they are most transparent, but they apply equally to the other models. First if the marginal tariff $p(q)$ obtained from (3.11) is not decreasing for all $q_1 \leqslant q \leqslant q_0$, it violates our buyer behavior assumptions. However, as Stiglitz [11] and Goldman, et al [3] have noted in the monopoly context, such solutions can be modified in a straightforward way to obtain a decreasing marginal tariff that satisfies the original optimization problem. In our case, (3.10) and (3.11) are modified so that for some interval $[a^*, b^*]$ of q containing the nondecreasing segment, we have

$$\int_{a^*}^{b^*} \{p^*[1 - 1/ne_N^*(q)] - c(q)\}\, dq = 0;$$

$$e_N^*(q) = -(\partial N/\partial p)((p/N)|_{p=p^*};$$

$$p(a^*) = p(b^*) = p^*.$$

The situation is illustrated in Fig. 2, where the parameters a^*, b^* and p^* are to be determined based on the three relationships above.

If $c(q)$ is nondecreasing in some region of q then, since $p(q) \to c(q)$ as n increases, the marginal price $p(q)$ determined by (3.11) will eventually become nondecreasing as well, invoking the above modification. This will

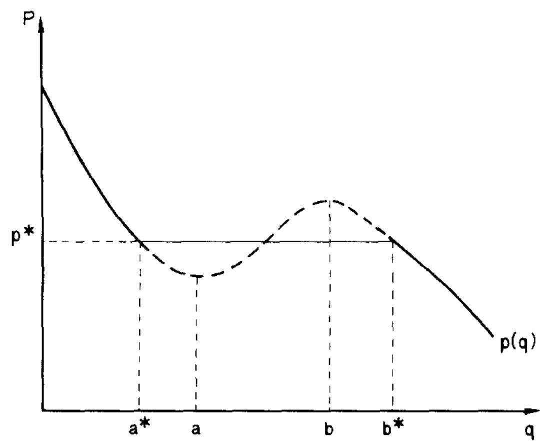

FIG. 2. Construction of monotone marginal tariff.

produce linear segments in the tariff precisely in the regions of q where $c(q)$ is decreasing, i.e., where total costs are convex. Furthermore, as $n \to \infty$, the marginal charge in the nondecreasing region of $c(q)$ becomes $p(q) \to p^* \to c(q^*)$, where q^* is the maximum purchase size demanded at p^*.

4. Analysis of Model VI

Again in our derivation we will suppress the index i. We note that $X(0) = 0$ (since the total demand for $t \leqslant 0$ is zero) and therefore $S(0) = 0$.

Integrating (2.13) by parts using (2.2) and Lemma 2.1 yields

$$NR[R; X] = [W(q_1; t_1) - C(q_1)][t_1 - S(t_1)] \tag{4.1}$$

$$- \int_0^{t_1} [\tau - S(\tau)][w(q^*(\tau); \tau) - c(q^*(\tau))][dq^*(\tau)/d\tau]\, d\tau,$$

$$\text{where} \quad q^*(t) = X'(t)/S'(t). \tag{4.2}$$

Replacing $q^*(t)$ and $dq^*(t)/dt$ in (4.1) by the values implied by (4.2) yields a formula for the firm's revenue in which the explicit choice available to the firm is the function S. [Note that this construction of the revenue formula, the simplest we have found, still depends on both the first and second derivatives of S.] The selection of $S(t)$ will be done using Euler's conditions while assuming that the boundary conditions q_1 and t_1 are fixed. These will be determined afterwards from transversality conditions. Suppressing the argument t in $q^*(t)$, $X(t)$ and $S(t)$, Euler's necessary condition for an optimal unconstrained choice of S can be written as[4]

$$[w(q; t) - c(q)]\, dq/dt$$
$$+ d\{(q^2/X')\, \partial\{[t - S][w(q; t) - c(q)]\}/\partial t\}/dt = 0. \tag{4.3}$$

[4] To obtain Eq. (4.3), we use the Euler condition

$$\partial f/\partial S - d\{\partial f/\partial S' - d[\partial f/\partial S'']/dt\}/dt = 0,$$

where by (4.2)

$$f(t, S, S', S'') = -[t - S][w(q; t) - c(q)][dq/dt],$$

with $q = X'/S'$ and $dq/dt = [X'' - S''q]q/X'$. After proper substitutions and suppression of the arguments in $w(q; t)$ and $c(q)$, this yields:

$$(w - c)\, dq/dt + d\{[q^2/X'][\partial f/\partial q] + d[(t - S)(w - c)\, q^2/X']/dt\}/dt = 0$$

Evaluating the derivatives in the second term above and collecting terms yields

$$(w - c)\, dq/dt + d\{[q^2/X'][(w - c)(1 - S') + [(t - S)\, \partial(w - c)/\partial t]]\}/dt = 0,$$

which reduces to (4.3).

Using the equal market share condition $S = (n - 1/n)t$, the symmetric Cournot equilibrium can then be characterized by the relation

$$[n-1][w(q;t) - c(q)]\, dq/dt + d\{q\partial\{t[w(q;t) - c(q)]\}/\partial t\}/dt = 0. \quad (4.4)$$

Integrating (4.4) with respect to t from 0 to t and recalling that $w(q^*, t) = R'(q^*)$, yields

$$q\{w(q;t) - c(q) + (t/n)\,\partial w(q;t)/\partial t\}$$
$$+ (1 - 1/n)\{R(q) - C(q) - q[R'(q) - c(q)]\} = K. \quad (4.5)$$

The constant of integration, K, is determined from the transversality conditions discussed in Appendix A. We remark, however, that $K = 0$ in the monopoly case $n = 1$. In that case, (4.5) becomes identical to condition (3.4) obtained for Model I. If the cost c is concave then $K/n \to 0$ and $R(q) \to C(q)$ as $n \to \infty$, satisfying the zero-profit condition. For intermediate values of n, (4.5) defines a function $t^*(q)$ which satisfies the first order conditions for a maximum, and can be used in (3.5) to obtain the marginal price schedule. However, unlike in Model I, we can not guarantee here that this solution satisfies the condition $w(q; t^*(q)) \geqslant c(q)$ for all values of q in the range of purchase sizes. When this condition does not hold, the integrand in (4.1) becomes negative and hence the maximum is achieved by choosing $t^*(q)$ so as to satisfy the equality

$$w(q;t) = c(q). \quad (4.6)$$

This implies that marginal cost pricing would be used in the ranges of q for which the solution to (4.5) would yield negative marginal net revenues.

Again we can alternatively express (4.5) in terms of the marginal price schedule $p(q)$ and the price elasticity of the aggregate demand function $N(p, q)$ introduced in Section 3, to obtain

$$\left\{p(q)\left(1 - \frac{1}{ne_N(q)}\right) - c(q)\right\}$$
$$+ (1 - 1/n)\left[\frac{R(q) - C(q)}{q} - (p(q) - c(q))\right] - \frac{K}{q} = 0. \quad (4.6)$$

The optimal marginal price schedule will then be given by $\max[p(q), c(q)]$.

Conditions (4.5) and (4.6) obtained for Model IV are identical in the monopoly case to the respective conditions (3.5) and (3.10) obtained for Model I. Models I and IV are also identical if the size distribution of purchases has the constant elasticity $e_N(q) = e$, in which case the schedule of marginal prices is $p(q) = c(q)/[1 - 1/ne]$, corresponding to a fixed profit margin $1/ne$ as a fraction of the price.

As to the range of purchase sizes; we already know that for the monopoly case, $n = 1$, and the perfect competition case $n \to \infty$, the last unit of the largest purchase size q_0, is priced at marginal cost and determined by the equation $w(q_0; 0) = c(q_0)$. One might expect that the tariffs for intermediate values of n will be bounded by these two extreme cases so that the largest purchase will be the same as above and the last unit will be priced at marginal cost for all values of n.

The smallest purchase size q_1 and the corresponding marginal subscriber's index t_1 are obtained from the transversality conditions, which are given in Appendix A. These conditions have force only if there is a fixed cost of supply. That is, if there is no fixed cost, $[C(q) \to 0$ as $q \to 0]$ then the minimum purchase size is $q_1 = 0$, and the corresponding fixed fee in the tariff satisfies $R(q_1) = 0$. When a fixed cost of supply is present (as is the case whenever a maintenance cost, packaging cost, or delivery cost is involved) the transversality conditions manifest an important economic aspect of competitive tariffs. They determine the fixed fee, or minimum subscription charge, obtained from any buyer making a purchase. This fee affects directly the range of purchase sizes and the number of active buyers, since those who would make small purchases in the absence of a fee are deterred by the fee from making any purchase at all. The optimal fixed fee and the minimum purchase size will both be positive if there is a fixed component of the supply costs.

5. An Illustrative Example

We illustrate some of the results by applying them to an example in which a buyer of type t has a willingness to pay function

$$\begin{aligned} w(q; t) &= 1 - q/(1-t), \qquad 0 < q < 1-t; \\ w(0; 1) &= 1. \end{aligned} \tag{5.1}$$

Thus, the total willingness-to-pay of buyer type t for quantity q is

$$W(q; t) = q - q^2/[2(1-t)]. \tag{5.2}$$

The corresponding aggregate demand function $N(p, q)$ is given by the value t for which $w(q; t) = p$, i.e.,

$$N(p, q) = 1 - q/[1-p], \qquad p < 1 - q. \tag{5.3}$$

On the supply side we specify that $C(q) = k + cq$, where $k \geqslant 0$ and $c \geqslant 0$.

Model I.

From condition (3.11) one obtains directly the schedule of marginal prices:

$$p(q) = 1 + ((n-1)/2)\,q - [((n-1)/2)^2\,q^2 + n(1-c)\,q]^{1/2}. \quad (5.4)$$

The largest purchase that would be made at cost is $(1-c)$, and we note from (5.4) that $p(1-c) = c$. Thus, the minimum purchase size q_1 is determined by condition (3.8), which together with (5.2) and (5.3) yields

$$R_1 = k + cq_1 + (1 - p(q_1) - q_1)(1 - p(q_1))/2n, \quad (5.5)$$

and by Lemma 2.1 which yields

$$R_1 = q_1[1 + p(q_1)]/2. \quad (5.6)$$

Together with (5.4), these two equations determine the minimum purchase q_1 and the corresponding tariff $R_1 = R(q_1)$. Remarkably, one finds that $q_1 = 2k/[1-c]$ independently of the number n of sellers.

As required for our formulation, the tariff is increasing and concave, and the schedule of marginal prices intersects each type's willingness-to-pay curve once from below. One can also use L'Hôpital's rule to show that $p(q) \to c$ and $R(q) \to C(q)$ as $n \to \infty$; that is, as required in the perfect competition case.

The resulting optimal tariff given by (3.7) is in this case

$$\begin{aligned} R(q) = R_1 &+ q + q^2(n-1)/4 \\ &- [q/2 + n(1-c)/(n-1)^2][1 - p(q) + q(n-1)/2] \\ &- (n^2(1-c)^2/(n-1)^3)\log(1 + [(n-1)/n][1 - p(q)]/[1-c]). \end{aligned} \quad (5.7)$$

Model VI

From the Euler condition (4.4) at a Cournot equilibrium, one obtains a partial characterization of the tariff, which in this example reduces to

$$R(q) = k + cq + [K + [1 - p(q)]^2 - q(1-c)]/(n-1), \quad (5.8)$$

where

$$p(q) = 1 - q/(1-t). \quad (5.9)$$

The constant K can be evaluated in terms of the smallest purchase size q_1 and corresponding customer index t_1. Using Lemma 2.1 and (5.2), (5.8) implies

$$K = nq_1(1-c) - (k + q_1 a/2)(n-1) - a^2, \quad (5.10)$$

where

$$a = q_1/(1 - t_1).$$

The smallest purchase size q_1 and marginal customer index t_1 can be calculated from the transversality conditions (A.2) and (A.4). In this example (A.2) becomes

$$n(1 - c)\, q_1 - \tfrac{1}{2}(n + 1)\, q_1 a - (n - 1)\, k = 0; \qquad (5.11)$$

while (A.4) reduces to

$$(n + 1)(1 - c)\, q_1 - \tfrac{1}{2}[n + 1]\, q_1 a - nk - \tfrac{1}{2}a^2 - t_1(1 - c - a)\, q'(t_1) = 0. \quad (5.12)$$

Using (5.11) we can simplify (5.12) to

$$q_1(1 - c) - k - \tfrac{1}{2}a^2 = t_1(1 - c - a)\, q'(t_1). \qquad (5.13)$$

The derivative $q'(t_1) = dq(t_1)/dt$ can be obtained from the Euler condition (4.4) which in this case reduces to

$$(1 - t_1)[n(1 - c - a) - a]\, q'(t_1) = 2a^2. \qquad (5.14)$$

Eliminating $q'(t_1)$ from (5.13) and (5.14) gives

$$(1 - t_1)[n(1 - c - a) - a][q_1(1 - c) - k - \tfrac{1}{2}a^2] - 2t_1 a^2(1-c-a) = 0. \quad (5.15)$$

Equations (5.11) and (5.15) can be solved for t_1 and q_1, and these can be used to determine K. Upon substitution of (5.11) into (5.10), the expression for K simplifies to the form

$$K = -t_1 a^2. \qquad (5.16)$$

The values of q_1 and t_1 also determine through (5.9) the marginal price for the q_1^{th} unit given by

$$p(q_1) = 1 - q_1/(1 - t_1). \qquad (5.17)$$

To determine the complete price schedule $p(q)$, it is more convenient to integrate directly (4.4) (after replacing $w(q^*; t)$ with $p(q)$). The marginal price schedule $p(q)$ is then obtained as the solution of the equation

$$\begin{aligned}&[2/(n - 1)]\{-(1 - p(q) + (n/(n - 1))(1 - c)\\ &\quad \times [1 - \log((n/2)(1 - c) - ((n - 1)/2)(1 - p(q)))]\} = q + L. \quad (5.18)\end{aligned}$$

The constant of integration L is determined by the boundary condition (5.17). The tariff $R(q)$ can then be obtained by substituting the marginal schedule $p(q)$ in the right hand side of (5.8). As pointed out earlier the schedule $p(q)$ is optimal only in the range of q where $p(q) \geqslant c$. Otherwise, we revert to marginal cost pricing. Hence, the complete price schedule is given by $\max[p(q), c]$. The last unit of the largest purchase size is again priced at marginal cost leading to $q_0 = 1 - c$.

In the monopoly case the solution t_1 is calculated directly from $q(t) = [1-c][1-t]^2$ to be $t_1 = 1 - [2k]^{1/2}/[1-c]$. If $n = \infty$, then $p(q) = c$ uniformly and $q(t) = (1-t)(1-m)$, so $t_1 = 1 - 2k/[1-c]^2$, and $q_1 = 2k/[1-c]$, and then $R(q_1) = k[1+c]/[1-c]$; thus, $R(q) = k + cq$ as anticipated.

To obtain a more concrete comparison between models I and IV we consider the special case $k = 0$, i.e., no fixed cost to the suppliers. For this case, the smallest purchase size is $q_1 = 0$ and $t_1 = 1$ in both models, and consequently $R(0^+) = 0$, $p(0) = 1$, and the constant of integration is $K = 0$. We define the discount function $D(q) = p(0)\, q - R(q)$ describing the difference between the nonlinear tariff and a linear tariff at the first unit price. Then $D'(q) = p(0) - p(q)$ and (5.4) can be expressed as

$$\text{(Model I)} \quad D'(q) = [n(1-c)\, q + ((n-1)\, q/2)^2]^{1/2} - (n-1)\, q/2. \tag{5.19}$$

Similarly (5.8) can be written now as

$$\text{(Model VI)} \qquad D'(q) = [n(1-c)\, q - (n-1)\, D(q)]^{1/2}. \tag{5.20}$$

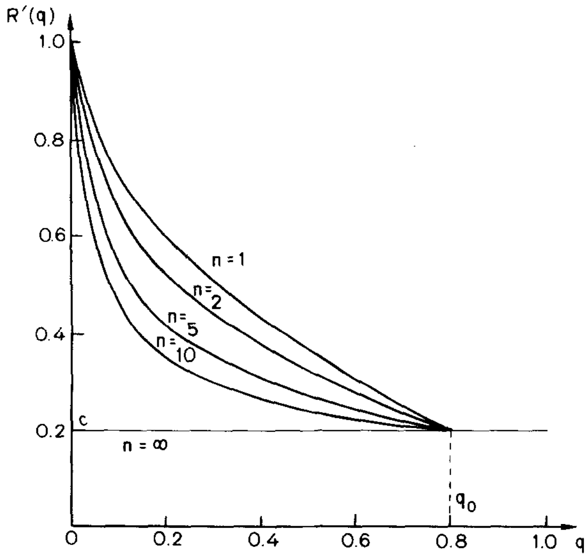

FIG. 3. Marginal price schedule for model I.

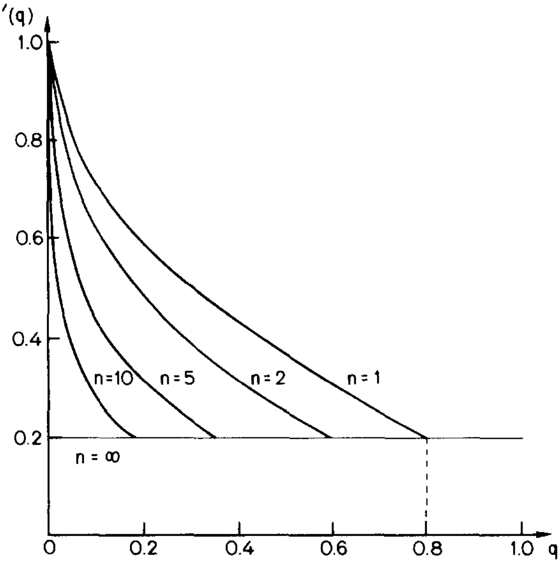

FIG. 4. Marginal price schedule for Model IV.

The marginal price schedules $R'(q)$ obtained in the two models for various values of n are illustrated in Figures 3 and 4. In both cases the initial condition is $D(0) = 0$. Clearly, the two models coincide in the monopoly case ($n = 1$), yielding $D'(q) = [(1-c)\,q]^{1/2}$, and as shown earlier for $n \to \infty$ $D'(q) = (1-c)$ in both cases. Also the maximum purchase size $q_0 = 1 - c$ for all n, in both cases, and the last unit of q_0 is priced at the marginal cost c. We note, however, that in Model I, for $q < q_0$, the marginal price $R'(q) = 1 - D'(q) > c$ for any finite n. Thus, the tariff in this case approaches perfect competition pricing from above, across the entire range of purchase sizes. On the other hand, in Model VI, since $D(1-c) \leqslant (1-c)^2$, (5.20) implies that for $n > 1, D'(q^*) = 1 - c$ for some $q^* < 1 - c$, so $R'(q) = c$ for a range of purchase sizes $[q^*, q_0]$, which expands as n increases. Thus, the perfect competition pricing is approached by expanding the range of purchase quantity that is priced at marginal cost until q^* equals the smallest purchase size q_1.

6. CONCLUSION

The primary aim of this paper has been to extend the study of Cournot models of competition to markets in which nonlinear prices prevail. In so doing, we have identified six different Cournot formulations that might be analyzed to obtain competitive equilibria. Two of the models (I and II) were shown to be equivalent, while Model V reduced to monopoly pricing with each supplier servicing the same customers regardless of price. Model I was the most amenable to a complete analysis. Here the equilibrium tariff was characterized by the classical Cournot elasticity condition, holding simultaneously at each possible purchase quantity level. Optimizing the boundary conditions allows the optimal tariff to be completely specified, as illustrated by the example. As a function of n, the number of suppliers, the results vary smoothly from the monopoly case $n = 1$ to the perfectly competitive case $n = \infty$, as illustrated in Fig. 3 and 4.

Model VI was also analyzed but produced less compact results. The transversality conditions allow the boundary values for its equilibrium tariff to be specified as well, so that a complete solution can be obtained here as well. The solution of the previous example allows the results to be compared with Model I. The remaining models, while necessary conditions for their equilibria can be obtained, provide little additional insight.

In the examples, the suppliers' equilibrium net revenue decreased for all quantity levels in moving from the monopoly case (Model V) to Model I to Model VI. The difference between these models can be characterized by how specific they are in describing each firm's market share. In Model V, certain customers are allocated to the competition regardless of price, leading to a monopoly pricing condition for each firm. Model I is less specific in its description, and Model VI is still less specific. In a Cournot equilibrium, increasing the ambiguity in describing the static behavior of one's competitors creates additional perceived opportunities for making inroads into their customer base through price cutting. Thus the equilibrium is pushed closer to the purely competitive case.

As a general matter it is desirable that economic theory undertake the study of nonlinear tariffs, if only to cope with the casual observation that they occur commonly in practice. In instances with which we are familiar, buyers are offered a choice among several two-part tariffs. The combined effect is a nonlinear tariff, obtained as the envelope of the two-part tariffs. The fact that these pricing practices are sustained in real-life oligopolies adds a measure of relevance to the formulations offered here.

APPENDIX

The following derivation of transversality conditions follows the methods presented by L. Elsgolc [1]. Applying the method of Elsgolc [1, pp. 81 and 100] for a free variation of $S(t_1)$ to the revenue formula implied by (4.1) and (4.2), yields the transversality condition

$$W(q_1; t_1) - C(q_1) - q_1[w(q_1; t_1) - c(q_1)][1 - q_1/X'(t_1)] = 0 \quad (A.1)$$

provided $n > 1$ (the corresponding variation of $q(t_1)$ in the monopoly case yields a vacuous condition corresponding to one less constant of integration in the Euler condition). At a Cournot equilibrium, $X'(t_1) = q_1(n-1)/n$, which reduces (A.1) to the form

$$W(q_1; t_1) - C(q_1) + q_1[w(q_1; t_1) - c(q_1)]/(n-1) = 0. \tag{A.2}$$

A second transversality condition is obtained by the free variation of t_1 yielding

$$\begin{aligned} &W(q_1; t_1) - C(q_1) + q_1[w(q_1; t_1) - c(q_1)] \\ &\qquad \times \{1 - X'(t_1)/q_1 - [t - S(t_1)]\, X''(t_1)/X'(t_1)\} \\ &\qquad + [t - S(t_1)]\, \partial W(q_1; t_1)/\partial t = 0. \end{aligned} \tag{A.3}$$

At a Cournot equilibrium this reduces to

$$\begin{aligned} &n[W(q_1; t_1) - C(q_1)] + [w(q_1; t_1) - c(q_1)] \\ &\qquad \times [q_1 - t_1\, dq(t_1)/dt_1] + t_1 \partial W(q_1; t_1)/\partial t_1 = 0, \end{aligned} \tag{A.4}$$

where one obtains $dq(t_1)/dt_1$ from the interior condition (4.5). The transversality conditions (A.2) and (A.4) determine t_1 and q_1. Since (4.5) determines $t^*(q)$ up to the constant K, this allows us to determine K. The tariff can then again be determined by (3.6). In the monopoly case, condition (A.4) is replaced by a condition including only the first and last terms in (A.4). This condition is identical to condition (3.9) obtained in Model I. Since in this case $K = 0$, this condition together with (4.5) completely determine t_1 and q_1. Again we note that if $n \to \infty$, then (A.4) yields $W(q_1; t_1) = C(q_1)$, which by (2.6) implies $R(q_1) = C(q_1)$, and, together with the Euler condition, yields $R(q) = C(q)$ for all $q \geqslant q_1$.

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