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Chapter 5

Hybrid Bertrand-Cournot Models of Electricity Markets with Multiple Strategic Subnetworks and Common Knowledge Constraints

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Abstract

Most existing Nash-Cournot models of competition among electricity generators assume that firms behave purely Cournot or Bertrand with respect to transmission decisions by the independent system operator. Such models are unrealistic for markets in which interfaces connecting subnetworks are frequently saturated but the congestion pattern within individual subnetworks is less predictable. We propose two approaches for dealing with such situation. The first is a hybrid Bertrand-Cournot model of these markets in which firms are assumed to behave *a la* Cournot regarding inter-subnetwork transmission quantities, but *a la* Bertrand regarding intra-subnetwork transmission prices. A second approach is a Bertrand type model where transmission lines that are congested most of the time are designated as "common knowledge constraint" and treated as equality constraints by all market participants including the ISO and all generation firms. Under affine demand functions and quadratic costs, the market equilibrium of these models becomes mixed linear complementarity problems with bisymmetric positive semi-definite matrices. Numerical examples demonstrate that dividing the network into strategic subnetwork leads to prices higher than those predicted by the pure Bertrand model, but lower than those from the pure Cournot model. When public knowledge constraints are recognized in a Bertrand type model the resulting equilibrium does not show a uniform change in prices relative to the pure Bertrand model but we observe a shift in output from lower cost to higher cost generators, lower prices at the high cost nodes and higher prices at the low cost nodes.

5.1 Introduction

Electricity market models are employed by market participants, policy markers, and stakeholders to characterize market agents' decisions and to predict market outcomes [21]. Many existing models follow game-theoretical approaches, but are distinguished by the formulation of the interaction between generators and the independent system operator (ISO). Such distinctions are referred to as sequential vs. simultaneous clearing of energy and transmission markets [1], or as integrated vs. separated market designs [16]. From a perspective of generator-ISO interaction, these models can be grouped into four approaches: Stackelberg, Stackelberg approximations, pure Cournot, and pure Bertrand.

The Stackelberg approach assumes that the energy and transmission markets are sequentially cleared, with generators acting first and the ISO acting second. Thus, generators anticipate the impact of their strategies on transmission prices (equivalent to locational price differences when the ISO computes nodal prices). The resulting model is a multi-leader one-follower Stackelberg game [2], [3], [9], [16]. Mathematically, a producer's decision problem is a mathematical program with equilibrium constraints (MPECs [14]), and the market equilibrium is an equilibrium problem with equilibrium constraints (EPEC [6]). However, this model introduces two difficulties arising from the embedded optimality conditions for the ISO's problem in all the generation firms' problems. First, the game among generators is a generalized Nash equilibrium problem [6] because each firm's decision variables appear in the constraint sets of the other firms' problems. Second, the firms' sets of feasible decisions are non-convex. As a result, this model may lead to either zero or multiple pure-strategy equilibria (see, for example, [2]). Moreover, even if a solution is found, it may be degenerate; that is, firms will find it optimal to barely congest some transmission lines so as to avoid congestion rents (see [17]; counter examples are given in [19]). Finally, finding equilibria of this model, even if one exists, for a realistic size network is computationally challenging.

Some approximations of the Stackelberg game have also been proposed that are computationally more tractable. One proposal assumes that generators hold fixed a priori conjectures concerning the sensitivity of transmission costs with respect to changes in amounts of transmission services requested [11] [1]. This model is formulated as a complementarity problem rather than a more difficult EPEC. However, exogenous response coefficients are unsatisfactory theoretically in that such responses should be the result of a game not an input, and problematic from an empirical viewpoint. Another approach [1] iterates between an ISO model and a generator model. The ISO model calculates sensitivities of transmission prices with respect to injections, and the generator models then calculate a Cournot equilibrium among generators, assuming that those sensitivities are constant. Given that equilibrium, the ISO model then checks if the same sensitivities indeed still hold; if not (which can happen if the set of binding transmission limits changes), new sensitivities are obtained, and passed back to the generators' models. However, this approach often fails to converge [1] [16].

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[1] The hybrid model of this paper can be viewed as an extreme case of the conjectured price model in [10] in which the slope of the transmission price with respect to changes in flows is either zero (Bertrand) or infinite (Cournot).

The Nash approaches assume that the energy and transmission markets are cleared simultaneously [8], [9], [15], [18], [21], [23] [24], [25]. In these approaches, generation firms do not explicitly model transmission limits in their constraint sets, and the ISO becomes a Nash player acting simultaneously with the firms. The market equilibrium is determined by aggregating the optimality conditions for the firms' and the ISO's problems, which become a mixed complementarity problem or (quasi-)variational inequalities. This approach avoids the computational intractability of the Stackelberg approaches and, under a non-degeneracy assumption, can lead to a unique equilibrium.

One variant of the Nash approach is the pure Cournot representation of generator expectations of ISO actions. The Cambridge-I model in [16] and the spot market model in [23] and [24] fall in the above category. In these models, firms are assumed to behave *a la* Cournot with respect to the ISO's, or arbitrageurs', that is, they treat as given the ISO's imports/experts into a bus or region, and act monopolistically with respect to the residual demand they face which are the horizontally shifted local demand curves. Such a pure Cournot model may be suitable for networks with relatively small interfaces between large markets that are frequently congested, especially for radial links such as the UK-France line. However, it is less realistic for general networks where generators may anticipate the impact of their outputs on interregional flows. Moreover, this model has the undesirable property that generators owned by one firm but located in different markets cannot coordinate decisions to their benefit; as a result, a company with plants in n different markets behaves the same as n separate firms (see Subsection 5.3.1.1 for more analysis).

Another variant of the Nash approach is the pure Bertrand model, in which generators take as given the nodal price markups due to congestion, or the locational price differences set by the ISO as congestion charges. In bilateral markets, this amounts to price taking behavior by generation firms with the respect to transmission services. In a POOLCO-like market, this model assumes that the residual demand function considered by generation firms take locational price differences as given by account for the fact that they can move all the prices up and down through their output decisions. As examples, Wei and Smeers [21] consider a Cournot game among generators with regulated transmission prices and solve a variational inequality problem to determine unique long-run equilibria. Smeers and Wei [18] consider a separated energy and transmission market, and show that such a market converges to the optimal dispatch with many marketeers. Hobbs et al. [8] [15] present Cournot equilibria in both bilateral and POOLCO markets with affine demand and cost functions, with the models formulated as mixed linear complementarity problems. Hobbs and Pang [9] formulate a bilateral market with piece-wise linear demand as a linear complementarity problem with a co-positive matrix. The model of spot wholesale markets developed in [23] and [24] takes into consideration the financial settlements of forward contracts.

Price taking in transmission is a defensible assumption for highly meshed networks that have several players and variable patterns of congestion. However, it is also naive as swing generators would probably try to influence locational price differences in their favor. Although, unlike the Cournot model, the Bertrand model allows a firm with plants in several locations to profitably coordinate decisions, it too has an undesirable property often referred to as the "thin line phenomenon". Under the Bertrand approach, to transmission, establishment of a thin line (say 1 MW) between two previously unconnected markets causes the model to treat the two markets as strategically linked which resulting in a much more competitive outcomes in both markets. For instance, two symmetric monopoly markets connected by such a thin line will result in a duopoly solution while the line carries zero flow.

Real power markets often consist of multiple subnetworks. In these markets, subnetworks are connected with frequently saturated interfaces, and hence they are decoupled in terms of strategic interaction since the residual demand functions in each subnet is shifted horizontally but their slopes (and hence elasticity) stays the same. On the other hand, congestion pattern within individual subnetworks is less predictable and hence generators within the subnetwork interact strategically. For instance, in Northwest Europe, the France-UK, France-Belgium, and Netherlands-Germany interconnections are usually congested, effectively decoupling the markets. A Cournot conjecture assuming that generators take interregional imports/exports as given regarding is reasonable in those cases. However, within the UK, German, and Benelux submarkets, congestion occurs but is less easily predicted, and in that case the Bertrand conjecture were generation firms behave as price takers with regard to transmission prices is more defensible. Therefore, neither the pure Cournot nor Bertrand models would be appropriate for markets multiple subnetworks. Even when the network cannot be divided into distict subnetworks, some transmission lines are systematically congested and such congestion is anticipated by all market participants who can predict how much power will flow across such interfaces and account for that in their strategic interaction. We refer to such transmission lines “common knowledge constraints” which can be accounted for within the Bertrand framework.

In this paper, we first consider exogenous subnetwork structures and propose a hybrid Bertrand-Cournot model that represents generators’ decision making in the presence of multiple subnetworks. This model assumes that firms behave *a la* Cournot with respect to the ISO’s inter-subnetwork transmission quantities, but *a la* Bertrand with respect to intra-subnetwork transmission prices. We then formulate a Bertrand type equilibrium model with certain links designated exogenously as common knowledge constraints.

The remainder of this paper is organized as follows. In the next section, we introduce the ISO's problem. Section 5.3 analyzes the shortcomings of the pure Cournot and pure models of generator-ISO interactions, and proposes a new hybrid model with multiple subnetworks; this section concludes with some results concerning solution uniqueness and computability. Section 5.4. reports numerical results and economic insights for the hybrid Bertrand-Cournot model applied to a stylized six node network. In Section 5.5 we introduce the formulation of the Bertrand model with public knowledge constraints and in Section 5.6 we apply this approach to a variant of the six node example introduced earlier. Concluding remarks are provided in Section 5.7.

5.2 The Role of the ISO

Electricity restructuring in different markets has followed several different blueprints [19]. In the US, one prevailing design is for the ISO to maintain a pool as a broker or auctioneer for wholesale spot transactions. In addition, the ISO controls the grid and transmits power from generators to consumers while meeting network and security constraints. The ISO also sets locational energy prices and transmission charges for bilateral energy transactions.

We consider an electricity network that is composed of nodes 1..*N* and transmission lines 1..*L*. This market consists of *G* competing firms, each firm *g*=1..*G* operating the units at $N_g \subseteq \{1..N\}$. We assume, without loss of generality, that there is one generation unit at each node: a demand-only node is denoted by a node with a zero-capacity generation unit, and a node with multiple units is split into multiple nodes.

Following the firms' decisions $\{q_i\}_{i=1}^N$, the ISO decides on nodal imports/exports $\{r_i\}_{i=1}^N$ that must obey the following constraints. Firstly, power flows should not exceed thermal or other limits $\{\bar{k}_i\}_{l=1}^L$ of transmission lines in both directions. We use a lossless DC approximation of Kirchhoff's laws (see [4]) and define power flows in terms of the so-called power transfer distribution factors (PTDFs). Each PTDF D_{li} represents a proportion of the flow occurring on the line *l*=1..*L* resulting from an one-unit injection of electricity at the node *i*=1…*N* and a corresponding one-unit withdrawal at the reference bus. These network feasibility constraints are

$$-\bar{k}_l \le \sum_{i=1}^N D_{li} r_i \le \bar{k}_l \ , \quad l = 1..L$$

Secondly, because electricity is not economically storable, load and generation must be balanced at all times. This establishes an energy balancing constraint, which sets the total import/export in a lossless grid to zero:

$$\sum_{i=1}^N r_i = 0$$

The non-storability of electricity also implies that the load at all nodes must be non-negative. Hence, the following constraints should also be considered in the ISO's decisions:

$$0 \le r_i + q_i \ , \quad i = 1..N \tag{5.1}$$

The objective of the ISO's transmission has been phrased as profit maximization in [8] and [10], cost minimization in [9] and social surplus maximization in [23], [24] and [25]. In this paper, we assume that the ISO aims to maximize social welfare, which denotes the total consumer willingness-to-pay, i.e., the area under the nodal inverse demand functions, less the total generation cost. The ISO is assumed to act *a la* Cournot with respect to generation injections, so the generation quantities are treated as given in its objective. Mathematically, the ISO's decision problem is

$$\max_{\{r_i\}_{i=1}^N} \sum_{i=1}^N \left((\int_0^{r_i+q_i} P_i(\tau_i) d\tau_i) - C_i(q_i) \right)$$

subject to:

$$\sum_{i=1}^N r_i = 0$$

$$\sum_{i=1}^{N} D_{li} r_i \geq -\bar{k}_l \,, \quad l = 1..L$$

$$\sum_{i=1}^{N} D_{li} r_i \leq \bar{k}_l \,, \quad l = 1..L$$

$$r_i + q_i \geq 0, \quad i = 1..N$$

Let p, λ_l^-, λ_l^+ and η_i be the Lagrange multipliers corresponding to the constraints, then the first order necessary optimality conditions (the Karush-Kuhn-Tucker, KKT conditions) for the ISO's problem are

- with respect to r_i

$$P_i(r_i + q_i) - p + \sum_{l=1}^{L} (\lambda_l^- - \lambda_l^+) D_{li} + \eta_i = 0, \quad i = 1..N$$

- with respect to p

$$\sum_{i=1}^{N} r_i = 0$$

- with respect to λ_l^-

$$0 \leq \lambda_l^- \quad \perp \quad \bar{k}_l + \sum_{i=1}^{N} D_{li} r_i \geq 0\,, \quad l = 1..L$$

- with respect to λ_l^+

$$0 \leq \lambda_l^+ \quad \perp \quad \bar{k}_l - \sum_{i=1}^{N} D_{li} r_i \geq 0\,, \quad l = 1..L$$

- with respect to η_i

$$0 \leq \eta_i \quad \perp \quad r_i + q_i \geq 0\,, \quad i = 1..N$$

Here, the first KKT condition identifies two parts of nodal prices:

$$P_i(r_i + q_i) = p + \varphi_i, \quad i = 1..N \tag{5.2}$$

where $\varphi_i = -\sum_{l=1}^{L} (\lambda_l^- - \lambda_l^+) D_{li} - \eta_i$.

We can interpret p as the reference energy price (when the nonnegative load constraint is not violated at the reference bus, this is just the price at the reference bus) and $\{\varphi_i\}_{i=1}^{N}$ as node specific premiums. Consequently, the congestion charge for the bilateral transmission from node *i* to node *j* is $\varphi_j - \varphi_i$, and the total congestion charge in the system is $\sum_{i=1}^{N} \varphi_i r_i$.

The ISO's transmission flows may lead to total payment from load differing from the total payment to generation. Hogan showed in [12] that the difference is non-negative. In the following, we quantify this difference.

Proposition 1: The difference between the total payment from load and the total charge from generation is equal to the total congestion charge in the network.

Proof: The total payment from load is the consumptions charged at nodal prices:

$$\sum_{i=1}^{N} P_i(r_i + q_i)\cdot(r_i + q_i)\,,$$

and the total charge from generation is the production compensated at nodal prices:

$$\sum_{i=1}^{N} P_i(r_i + q_i)\cdot q_i\ .$$

Their difference, denoted by Δ, is

$$\Delta = \sum_{i=1}^{N} P_i(r_i + q_i)\cdot r_i$$

By condition (5.2), we have

$$\Delta = \sum_{i=1}^{N} \varphi_i \cdot r_i$$

and this difference is equivalent to the total congestion charge. Furthermore, Δ can be written as

$$\Delta = \sum_{l=1}^{L} (\lambda_l^- + \lambda_l^+)\,\bar{k}_l + \sum_{i=1}^{N} q_i \eta_i = 0$$

Because $\{\lambda_l^-\}_{l=1}^{L}$, $\{\lambda_l^+\}_{l=1}^{L}$, $\{q_i\}_{i=1}^{N}$ and $\{\eta_i\}_{i=1}^{N}$ are nonnegative, this difference is also nonnegative.

5.3 The Hybrid Subnetwork Model

5.3.1 Two Existing Models

Before introducing our model of multiple subnetworks, we review two existing Nash models and analyze their limitations. Both models assume the ISO to be Nash player, and present market equilibrium conditions by aggregating the KKT conditions for the firms' and the ISO's problems.

5.3.1.1 The Pure Cournot Model

The first model assumes that the firms behave purely Cournot with respect to the ISO's transmitted quantities (see, for example, [14], [23], [24), that is, they take as given the ISO's import/exports at each node. Hence, a firm g=1..G solves the following profit-maximization problem which is parameterized by $\{r_i\}_{i=1}^{N}$:

$$\max_{\{q_i\}_{i\in Ng}} \ \sum_{i\in Ng} P_i(r_i + q_i)\cdot q_i - \sum_{i\in Ng} C_i(q_i)$$

subject to:

$$q_i \geq 0\,, \quad i \in N_g$$
$$q_i \leq \bar{q}_i\,, \quad i \in N_g$$
$$r_i + q_i \geq 0\,, \quad i = N_g$$

Here, $\bar{q}_i$ and $C_i(q_i)$ are the capacity and cost function of the unit at node i, respectively.

Notice that firm g's problem can be decomposed into N_g subproblems, each corresponding to the firm's production decision at one node with a nodal demand function that has been shifted by the import by the ISO. As we noted earlier, this model will predict a market equilibrium that is not affected by whether or not generators in different locations are owned by the same firm. Moreover, under this formulation, the equilibrium solution for a network in which no transmission constraints are binding predicts average nodal prices that are higher than the Cournot equilibrium price corresponding to a single market with the aggregated system demand function. For example, in a two-node system with unlimited transmission capacity and with symmetric supply and demand (so that each node is self sufficient) , this model yields the monopoly price at each node since it does not account for the reduced demand elasticity resulting from the merging of the two local monopoly markets into a duopoly. For the special case of zero generation cost, the monopoly price resulting from the model is twice the duopoly price corresponding to the merged markets.

5.3.1.2 The Pure Bertrand Model

The second model assumes that the firms behave purely Bertrand with regard to the ISO's transmission prices [25]. This is achieved by rewriting (5.2) as

$$r_i + q_i = P_i^{-1}(p + \varphi_i), \quad i = 1..N$$

and summing it up for all nodes:

$$\sum_{i=1}^{N}(r_i + q_i) = \sum_{i=1}^{N} P_i^{-1}(p + \varphi_i)$$

Due to the energy balance constraint on the ISO redispatch we get:

$$\sum_{i=1}^{N} q_i = \sum_{i=1}^{N} P_i^{-1}(p + \varphi_i) \tag{5.3}$$

which, implicitly, characterizes the residual demand function faced by each generation firm given the locational markups set by the ISO.

Now, the firms' competition can be modeled as a Nash-Cournot game among generators where each firm takes as given its competitors' production as well the nodal price premiums $\{\varphi_i\}_{i=1}^{N}$, and acts as a monopolist with respect to the residual demand implied by (3). Mathematically, a firm g solves the following problem:

$$\max_{\{q_i\},\, p} \sum_{i \in Ng}(p + \varphi_i)q_i - \sum_{i \in Ng} C_i(q_i)$$

subject to:

$$q_i \geq 0\,, \quad i \in N_g$$

$$q_i \le \overline{q}_i\,, \quad i \in N_g$$
$$\sum_{i=1}^{N} q_i = \sum_{i=1}^{N} P_i^{-1}(p+\varphi_i)$$
$$r_i + q_i \ge 0\,, \quad i = 1..N$$

Because the firms observe the system-wide demand, their decision problems are not geographically separable and this model's solution is sensitive to whether or not a firm owns plants at different locations. Moreover, when the network constraints are not violated, the locational price premiums go to zero, and this model produces the same solution as a Cournot equilibrium calculated for a single node with the aggregated system demand. Unfortunately, this model has a shortcoming when applied to systems with multiple subnetworks. For instance, in the case of a two-node one-line network, reducing the line capacity to zero creates two local monopolies. However, this model will still yield a duopoly equilibrium with prices lower than the monopoly prices.

5.3.2 The Hybrid-Bertrand-Cournot Model

In the following, we introduce a model of Cournot competition among generators that is capable of separating the firms' decision making into strategic subnetworks. In doing so, we study the preceding two models, and observe that whether or not the firms' decision problems are separable depends on the slopes of the residual demand functions they face. In this new model, we assume firms behave *a la* Cournot with respect to inter-subnetwork imports/exports, but *a la* Bertrand to intra-subnetwork transmission costs. As a result, the firms' decisions in different subnetworks are essentially independent, but the generator ownership structures within individual subnetworks affect the solution.

5.3.2.1 The Firms' Problems

The first step to characterize the firms' problems is to quantify the aggregated demand function in each subnetwork. This is obtained by summing the inverse function of (5.2) for the node set $\tilde{N}_s$ of each subnetwork *s* (replacing *p* with p_s):

$$\sum_{i\in\tilde{N}_s} q_i + \sum_{i\in\tilde{N}_s} r_i = \sum_{i\in\tilde{N}_s} P_i^{-1}(p_s+\varphi_i), \qquad s = 1..S$$

This equation allows the firms to compete for sales in each subnetwork. In mathematical terms, the decision problem for a firm *g* is

$$\max_{\{q_i\}_{i\in N_g},\{p_s\}_{s=1}^{S}} \sum_{s=1}^{S} \sum_{i\in Ng\cap\tilde{N}_s} (p_s+\varphi_i)q_i - \sum_{i\in Ng} C_i(q_i)$$

subject to:

$$q_i \ge 0\,, \quad i \in N_g$$
$$q_i \le \overline{q}_i\,, \quad i \in N_g$$

$$\sum_{i \in \tilde{N}_s} q_i + \sum_{i \in \tilde{N}_s} r_i = \sum_{i \in \tilde{N}_s} P_i^{-1}(p_s + \varphi_i), \qquad s = 1..S$$

$$r_i + q_i \geq 0\,, \quad i = 1..N$$

Because this problem is parameterized by the total import/export $\sum_{i \in \tilde{N}_s} r_i$ in each subnetwork s and the locational price premiums $\{\varphi_i\}_{i=1}^N$, it can only be decomposed according to the structure of the subnetworks. If we let ρ_i^-, ρ_i^+, β_{gs}, and ξ_i be the Lagrange multipliers corresponding to the constraints, the KKT conditions for this problem are

- with respect to q_i:

$$p_s + \varphi_i - \beta_{gs} - C_i'(q_i) + \rho_i^- - \rho_i^+ + \eta_i = 0, \qquad i \in \tilde{N}_s \cap N_g,\ s = 1..S$$

- with respect to p_s:

$$\beta_{gs} \sum_{i \in \tilde{N}_s} \frac{dP_i^{-1}(p_s + \varphi_i)}{dp_s} + \sum_{i \in \tilde{N}_s \cap N_g} q_i = 0, \qquad s = 1..S$$

- with respect to β_{gs}

$$\sum_{i \in \tilde{N}_s} q_i + \sum_{i \in \tilde{N}_s} r_i = \sum_{i \in \tilde{N}_s} P_i^{-1}(p_s + \varphi_i), \qquad s = 1..S$$

- with respect to ρ_i^-

$$0 \leq \rho_i^- \ \perp\ q_i \geq 0, \quad i \in N_g$$

- with respect to ρ_i^+

$$0 \leq \rho_i^+ \ \perp\ \bar{q}_i - q_i \geq 0, \quad i \in N_g$$

- with respect to ξ_i

$$0 \leq \xi_i \ \perp\ r_i + q_i \geq 0, \quad i \in N_g$$

5.3.2.2 The Market Equilibrium Conditions

The market equilibrium conditions of the model are obtained by combining the KKT conditions for the ISO’s and the firms' programs. In general, these conditions form a mixed nonlinear complementarity problem. When the nodal demand functions are linear and the cost functions are convex quadratic, i.e.,

$$P_i(q) = a_i - b_i q, \qquad i = 1..N$$

$$C_i(q) = c_i q + \tfrac{1}{2} d_i q^2, \qquad i = 1..N$$

the market equilibrium conditions become the following mixed linear complementarity problem (mixed LCP, see [5]):

$$p_s + \varphi_i - \beta_{gs} - c_i - d_i q_i + \rho_i^- - \rho_i^+ + \eta_i = 0, \qquad i \in \tilde{N}_s \cap N_g,\ s = 1..S, \ \ g = 1..G$$

$$-\beta_{gs}\sum_{i\in\tilde{N}_s}\frac{1}{b_i}+\sum_{i\in\tilde{N}_s\cap N_g}q_i=0,\qquad s=1..S,\ g=1..G$$

$$\sum_{i\in\tilde{N}_s}q_i+\sum_{i\in\tilde{N}_s}r_i=\sum_{i\in\tilde{N}_s}\frac{a_i-p_s-\varphi_i}{b_i},\qquad s=1..S$$

$$0\le\rho_i^-\ \perp\ q_i\ge 0,\quad i\in N$$

$$0\le\rho_i^+\ \perp\ \bar{q}_i-q_i\ge 0,\quad i\in N$$

$$0\le\xi_i\ \perp\ r_i+q_i\ge 0,\quad i=1..N$$

$$P_i(r_i+q_i)-p+\sum_{l=1}^{L}(\lambda_l^- -\lambda_l^+)D_{li}+\eta_i=0,\quad i=1..N$$

$$\sum_{i=1}^{N}r_i=0$$

$$0\le\lambda_l^-\ \perp\ \bar{k}_l+\sum_{i=1}^{N}D_{li}r_i\ge 0,\quad l=1..L$$

$$0\le\lambda_l^+\ \perp\ \bar{k}_l-\sum_{i=1}^{N}D_{li}r_i\ge 0,\quad l=1..L$$

$$\eta_i\ \perp\ r_i+q_i\ge 0,\quad i=1..N$$

This problem is not a square LCP because different Lagrange multipliers (η_i and ξ_i) are assigned to the common constraints (5.1) shared by the ISO's and the firms' problems. However, from an economic point of view, it is reasonable to assume that in equilibrium these common constraints should have the same shadow values for each entity, that is,

$$\eta_i=\xi_i,\quad i=1..N$$

The practical importance of this, arguably strong assumption, is negligible, since the shadow prices are positive only if nodal prices are above the choke price on the demand curves, and the load is zero. This is a very unlikely occurrence in practice. Making this assumption is mathematically convenient, turning the market equilibrium conditions into the following square mixed LCP problem:

$$p_s+\varphi_i-\beta_{gs}-c_i-d_iq_i+\rho_i^- -\rho_i^+ +\eta_i=0,\qquad i\in\tilde{N}_s\cap N_g,\ s=1..S,\ g=1..G \tag{5.4}$$

$$-\beta_{gs}\sum_{i\in\tilde{N}_s}\frac{1}{b_i}+\sum_{i\in\tilde{N}_s\cap N_g}q_i=0,\qquad s=1..S,\ g=1..G \tag{5.5}$$

$$\sum_{i\in\tilde{N}_s}q_i+\sum_{i\in\tilde{N}_s}r_i=\sum_{i\in\tilde{N}_s}\frac{a_i-p_s-\varphi_i}{b_i},\qquad s=1..S \tag{5.6}$$

$$0\le\rho_i^-\ \perp\ q_i\ge 0,\qquad i=1..N \tag{5.7}$$

$$0\le\rho_i^+\ \perp\ \bar{q}_i-q_i\ge 0,\qquad i=1..N \tag{5.8}$$

$$a_i-b_i(q_i+r_i)-p-\varphi_i=0,\qquad i=1..N \tag{5.9}$$

$$\sum_{i=1}^{N}r_i=0 \tag{5.10}$$

$$0 \le \lambda_l^- \quad \perp \quad \bar{k}_l + \sum_{i=1}^{N} D_{li} r_i \ge 0, \quad l = 1..L \tag{5.11}$$

$$0 \le \lambda_l^+ \quad \perp \quad \bar{k}_l - \sum_{i=1}^{N} D_{li} r_i \ge 0, \quad l = 1..L \tag{5.12}$$

$$0 \le \eta_i \quad \perp \quad r_i + q_i \ge 0, \quad i = 1..N \tag{5.13}$$

$$\varphi_i = -\sum_{l=1}^{L} (\lambda_l^- - \lambda_l^+) D_{li} - \eta_i = 0, \quad i = 1..N \tag{5.14}$$

In Subsection 5.3.1.1, we introduced the reference prices $\{p_s\}_{s=1}^{S}$ of the subnetworks to construct the aggregated subnetwork demand functions in the firms' problems. Because the firms have the full ability to influence these reference prices, our model will be incorrect if these prices aren't equal at the equilibrium.

Proposition 2: In the market equilibrium, all reference prices are equal, that is

$$p_s = p, \quad s = 1..S$$

Proof: Condition (6) implies

$$p_s = \frac{\sum_{i \in \tilde{N}_s} \frac{a_i - \varphi_i}{b_i} - \sum_{i \in \tilde{N}_s} q_i - \sum_{i \in \tilde{N}_s} r_i}{\sum_{i \in \tilde{N}_s} \frac{1}{b_i}}, \qquad s = 1..S$$

Solving for $\{r_i\}_{i=1}^{N}$ from (9) and substituting the values into the above expression gives

$$p_s = \frac{\sum_{i \in \tilde{N}_s} \frac{a_i - \varphi_i}{b_i} - \sum_{i \in \tilde{N}_s} q_i - \sum_{i \in \tilde{N}_s} \left(\frac{a_i - p - \varphi_i}{b_i} - q_i \right)}{\sum_{i \in \tilde{N}_s} \frac{1}{b_i}} = p, \qquad s = 1..S$$

5.3.2.3 Computational Properties

The preceding market equilibrium conditions represent a quasi-variational inequality problem due to the common constraints in the firms' and the ISO's programs. Next, we study its solution existence and the solution approach.

Lemma 1: Conditions (5.4)-(5.14) can be represented as a linear complementarity problem with a bisymmetric positive semi-definite matrix.

Proof: We group the parameters and variables as follows:

- $B \in R^{N \times N}$: A diagonal matrix where the (i,i) th is b_i,
- $D \in R^{L \times N}$: The PTDF matrix where the (l,i) th element is D_{li},
- $a = [a_i \quad i = 1..N]$,
- $\bar{q} = [\bar{q}_i \quad i = 1..N]$,

- $c = [c_i \quad i = 1..N]$,
- $\bar{k} = [\bar{k}_l \quad l = 1..L]$,
- $r = [r_i \quad i = 1..N]$,
- $q = [q_i \quad i = 1..N]$,
- $\rho_- = [\rho_i^- \quad i = 1..N]$,
- $\rho_+ = [\rho_i^+ \quad i = 1..N]$,
- $\lambda_- = [\lambda_l^- \quad l = 1..L]$,
- $\lambda_+ = [\lambda_l^+ \quad l = 1..L]$,
- $\eta = [\eta_i \quad i = 1..N]$.

Further let $e \in R^N$ be a vector of all 1's, and $H \in R^{N \times N}$ and $Q \in R^{N \times N}$ be two matrices such that

$$h_{ij} = \begin{cases} \dfrac{1}{\sum_{i=1}^{N} \frac{1}{b_i}} + \dfrac{1}{\sum_{i \in \tilde{N}_S} \frac{1}{b_i}} + d_i & \text{if } \; i = j \\[2ex] \dfrac{1}{\sum_{i=1}^{N} \frac{1}{b_i}} + \dfrac{1}{\sum_{i \in \tilde{N}_S} \frac{1}{b_i}} & \text{if } \; i \neq j, \text{ and nodes } i \text{ and } j \text{ belong to the same subnetwork} \\ & \text{and the same firm} \\[2ex] \dfrac{1}{\sum_{i=1}^{N} \frac{1}{b_i}} & \text{Otherwise} \end{cases}$$

and

$$Q = B^{-1} - \frac{B^{-1} e e^T B^{-1}}{e^T B^{-1} e}.$$

Eliminating variables with free signs, we represent (5.4) – (5.14) as an LCP problem:

$$w = t + My \tag{5.15a}$$
$$0 \leq w \quad \perp \quad y \geq 0 \tag{5.15b}$$

where w and y are variable vectors, t and M are constants, such that

$$w = \begin{bmatrix} \rho_- \\ \bar{q} - q \\ \bar{k} + Dr \\ \bar{k} - Dr \\ r + q \end{bmatrix} \quad y = \begin{bmatrix} q \\ \rho_+ \\ \lambda_- \\ \lambda_+ \\ \eta \end{bmatrix} \quad t = \begin{bmatrix} c - \dfrac{ee^T B^{-1}}{e^T B^{-1} e} a \\ \bar{q} \\ \bar{k} + DQa \\ \bar{k} - DQa \\ Qa \end{bmatrix}$$

and

$$M = \begin{bmatrix} H & I & BQD^T & -BQD^T & BQ - I \\ -I & 0 & 0 & 0 & 0 \\ -DQB & 0 & DQD^T & -DQD^T & DQ \\ DQB & 0 & -DQD^T & DQD^T & -DQ \\ QB + I & 0 & QD^T & -QD^T & Q \end{bmatrix};$$

Note H and Q are both symmetric positive semi-definite and

$$\frac{M + M^T}{2} = \begin{bmatrix} H & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ D \\ -D \\ I \end{bmatrix} Q \begin{bmatrix} 0 \\ 0 \\ D \\ -D \\ I \end{bmatrix}^T$$

we conclude that M is bisymmetric positive semi-definite. □

Lemma 2: there exists at least one market equilibrium.

Proof: We observe that

$$w = \begin{bmatrix} c \\ \bar{q} \\ \bar{k} \\ \bar{k} \\ Qa \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a \end{bmatrix}$$

satisfy the linear conditions in (5.15). By Theorem 3.1.2 in [5], we conclude that conditions (5.4) – (5.14) have solutions. □

Theorem 1: Assuming non-degeneracy, a solution to (5.4) – (5.14) is guaranteed by Lemke's algorithm [11].

Proof: This follows Theorem 4.4.1 in [5] and lemmas 1 and 2. □

It is worth noting that the models introduced in Section 5.3.1 are two special cases of (5.4) – (5.14) with N and 1 subnetworks, respectively. Their equilibrium conditions can also be presented as (5.15) where H is given by

$$h_{ij} = \begin{cases} \dfrac{1}{\sum_{i=1}^{N} \dfrac{1}{b_i}} + b_i + d_i & \text{if } \; i = j \\ \dfrac{1}{\sum_{i=1}^{N} \dfrac{1}{b_i}} & \text{Otherwise} \end{cases}$$

for the extreme case where all N nodes are strategically decoupled and represent N subnetwork, and by

$$h_{ij} = \begin{cases} \dfrac{2}{\sum_{i=1}^{N} \dfrac{1}{b_i}} + d_i & \text{if } \; i = j \\ \dfrac{2}{\sum_{i=1}^{N} \dfrac{1}{b_i}} & \text{if } \; i \neq j, \text{ and nodes } i \text{ and } j \text{ belong to the same firm} \\ \dfrac{1}{\sum_{i=1}^{N} \dfrac{1}{b_i}} & \text{Otherwise} \end{cases}$$

for the case where all generators are strategically coupled in a single subnetwork.

5.4 Numerical Example for the Subnetworks Model

We use the network in Fig. 5.1 to illustrate the application and economics insights of the hybrid Bertrand-Cournot model. In this example, all eight lines are identical in terms of their electrical characteristics except that the two interfaces, lines 2-4 and 3-5, that have very low thermal limits of 2 MW. As a result, this network is separated into two strategic subnetworks where nodes 1 through 3 form one subnetwork and nodes 4 through 6 form the other. In addition, this system has six generators, each at one node (see Table 5.1). We assume four different hypothetical generator ownership structures (See Table 5.2) by assigning the generators to 2, 3, 4 and 6 firms, respectively; such structures enable us to observe the sensitivity of the market equilibrium to market concentration.

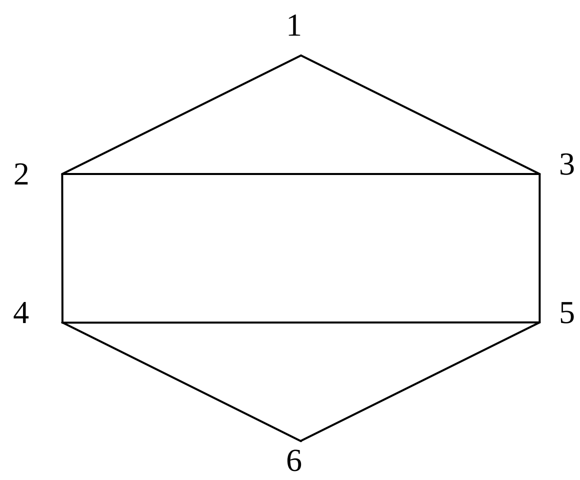

Fig. 5.1 A 6-Bus Network

Table 5.1 Generator information

Node	*Capacity* (MW)	*Marginal cost* ($/MWh)
1	120	15
2	80	20
3	25	30
4	80	20
5	25	30
6	120	15

We consider affine demand functions (see Table 5.3) which, together with the generator characteristics and the resource ownership patterns, lead to symmetric subnetworks. Therefore, as would be expected, the resulting nodal prices are uniform across all the nodes, and the flows on the interfaces are zero (see Table 5.4). In addition, our model predicts prices that are lower than the prices from the pure Cournot model (that treats each node as a subnetwork, so that generators believe that they face only the local price elasticity) and greater than those from the pure Bertrand model (that treats the entire network as a single subnetwork, resulting in generators believing that they can compete in all markets). It should also be pointed out that the hybrid Bertrand-Cournot model produces the same market outcomes for the three- and four-firm structures; this is because, in these structures, both subnetworks consist of duopoly firms.

Table 5.2 Generator ownership structures

Node	*2 firms*	*3 firms*	*4 firms*	*6 firms*
1	Firm #1	Firm #1	Firm #1	Firm #1
2	Firm #1	Firm #3	Firm #3	Firm #2
3	Firm #1	Firm #1	Firm #1	Firm #3
4	Firm #2	Firm #3	Firm #4	Firm #4
5	Firm #2	Firm #2	Firm #2	Firm #5
6	Firm #2	Firm #2	Firm #2	Firm #6

Table 5.3 Affine demand functions

Node	*Price intercept ($/MWh)*	*Slope*
1	100	1.0
2	100	0.8
3	100	1.2
4	100	0.8
5	100	1.2
6	100	1.0

Table 5.4: Nodal prices under symmetric subnetworks

Model	*Node*	*2 firms*	*3 firms*	*4 firms*	*6 firms*
Hybrid	1-6	60.00	45.00	45.00	42.30
Pure Cournot	1-6	60.63	60.63	60.63	60.63
Pure Bertrand	1-6	46.67	40.54	35.14	32.86

Table 5.5 Generator outputs under symmetric subnetworks

	Node	*2 firms*	*3 firms*	*4 firms*	*6 firms*
Hybrid Cournot-Bertrand	1	120.00	92.50	92.50	84.17
	6	120.00	92.50	92.50	84.17
	2	3.33	77.08	77.08	68.75
	4	3.33	77.08	77.08	68.75
	3	0	0	0	25.00
	5	0	0	0	25.00
Pure Cournot	1	45.63	45.63	45.63	45.63
	6	45.63	45.63	45.63	45.63
	2	50.78	50.78	50.78	50.78
	4	50.78	50.78	50.78	50.78
	3	25.00	25.00	25.00	25.00
	5	25.00	25.00	25.00	25.00
Pure Bertrand	1	120.00	120.00	120.00	110.12
	6	120.00	120.00	120.00	110.12
	2	44.44	63.33	80.00	79.29
	4	44.44	63.33	80.00	79.29
	3	0	0	0	17.62
	5	0	0	0	17.62

Next, we create an asymmetric structure of the subnetworks by exchanging the cost functions at nodes 2 and 3. Such asymmetry is more likely to result in flow congestion on the interfaces and uneven nodal prices. Indeed, this is found true for most test scenarios except the duopoly structure (see Table 5.6). Again, the hybrid Bertrand-Cournot model leads to prices between those from the pure Cournot and Bertrand models. Similar to the symmetric case, our model produces identical market equilibria for the three- and four-firm structures.

Table 5.6 Equilibria of asymmetric subnetworks

Model	*Node*	*Nodal Prices*				*Generator outputs*			
		2 firms	*3 firms*	*4 firms*	*6 firms*	*2 firms*	*3 firms*	*4 firms*	*6 firms*
Hybrid Cournot-Bertrand	1	60.00	47.88	47.88	45.18	120.00	101.39	101.39	93.05
	2	60.00	49.07	49.07	46.37	0	58.81	58.81	50.48
	3	60.00	46.69	46.69	43.99	3.33	0	0	25.00
	4	60.00	44.26	44.26	41.56	3.33	74.80	74.80	66.46
	5	60.00	46.64	46.64	43.94	0	0	0	25.00
	6	60.00	45.45	45.45	42.75	120.00	93.89	93.89	85.56
Pure Cournot	1	62.73	62.73	62.73	62.73	47.73	47.73	47.73	47.73
	2	63.11	63.11	63.11	63.11	41.39	41.39	41.39	41.39
	3	62.36	62.36	62.36	62.36	25.00	25.00	25.00	25.00
	4	60.48	60.48	60.48	60.48	50.61	50.61	50.61	50.61
	5	61.23	61.23	61.23	61.23	25.00	25.00	25.00	25.00
	6	60.86	60.86	60.86	60.86	45.86	45.86	45.86	45.86
Pure Bertrand	1	49.78	44.02	39.71	36.32	120.00	120.00	120.00	120.00
	2	54.78	47.50	40.60	38.20	17.32	44.43	65.34	50.56
	3	44.78	40.55	38.83	34.44	15.46	6.72	0	25.00
	4	45.23	37.50	34.43	31.30	34.03	63.46	80.00	69.70
	5	55.23	44.44	36.19	35.06	1.52	0	0	25.00
	6	50.23	40.97	35.31	33.18	120.00	120.00	120.00	112.11

5.5 Bertrand Model with Common Knowledge Constraints

In the pure Bertrand model deascribed in Section 5.3.1.2, it was assumed that firms optimizing their profit are not aware of any transmission constraints and account for congestion only through the nodal price markups set by the ISO to which the respond as price takers. In this section we will allow firms to directly account to congestion on lines that are systematically capacitated and hence are designated as common knowledge constraints, while they still act at price takers to the portion of the nodal price markups that reflect congestion on lines that have not been so designated.

5.5.1 The Firm's Problems

When some line sets, $L_1 \subseteq L$ and $L_2 \subseteq L$, are constantly congested in the negative and positive directions, respectively, the ISO's KKT conditions take the form:

$$P_i(r_i + q_i) - p + \sum_{l \in L_1} D_{li}\lambda_l^- - \sum_{l \in L_2} D_{li}\lambda_l^+ + \sum_{l \in L \setminus L_1} (\lambda_l^- - \lambda_l^+) D_{li} = 0, \quad i \in N \tag{5.16}$$

$$\sum_{i \in N} r_i = 0 \tag{5.17}$$

$$\bar{k}_l + \sum_{i \in N} D_{li} r_i = 0, \quad \lambda_l^+ = 0, \ \ l \in L_1 \tag{5.18}$$

$$\bar{k}_l - \sum_{i \in N} D_{li} r_i = 0, \quad \lambda_l^- = 0, \ \ l \in L_2 \tag{5.19}$$

$$0 \le \lambda_l^- \ \perp \ \bar{k}_l + \sum_{i \in N} D_{li} r_i \ge 0, \qquad l \in L \setminus L_1 \tag{5.20}$$

$$0 \le \lambda_l^+ \ \perp \ \bar{k}_l - \sum_{i \in N} D_{li} r_i \ge 0, \qquad l \in L \setminus L_2 \tag{5.21}$$

Let

$$\tilde{\varphi}_i = - \sum_{l \in L \setminus L_1 \setminus L_2} (\lambda_l^- - \lambda_l^+) D_{li} \,,$$

then condition (5.16) can be rewritten as

$$P_i(r_i + q_i) + p - \sum_{l \in L_1} D_{li}\lambda_l^- + \sum_{l \in L_2} D_{li}\lambda_l^+ + \tilde{\varphi}_i = 0, \quad i \in N \tag{5.22}$$

Thus, the nodal prices are composed of three parts: the price p at the reference bus, the locational price premium $-\sum_{l \in L_1} D_{li}\lambda_l^- + \sum_{l \in L_2} D_{li}\lambda_l^+$ due to the systematically congested constraints and the locational price markups $\tilde{\varphi}_i$ that account for all the other constraints and taken by firms as ISO set parameters when valuating their residual demand. Solving for r_i in (5.22) and substituting into (5.17), (5.18), and (5.19) yields:

$$\sum_{i \in N} \left(P_i^{-1} \left(p - \sum_{l \in L_1} D_{li}\lambda_l^- + \sum_{l \in L_2} D_{li}\lambda_l^+ + \tilde{\varphi}_i \right) - q_i \right) = 0 \tag{5.23}$$

$$\bar{k}_l + \sum_{i \in N} D_{li} \left(P_i^{-1} \left(p - \sum_{m \in L_1} D_{mi}\lambda_m^- + \sum_{m \in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i \right) - q_i \right) = 0, \qquad l \in L_1 \tag{5.24}$$

$$\bar{k}_l - \sum_{i \in N} D_{li} \left(P_i^{-1} \left(p - \sum_{m \in L_1} D_{mi}\lambda_m^- + \sum_{m \in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i \right) - q_i \right) = 0, \qquad l \in L_2 \tag{5.25}$$

When the systematically congested lines are common knowledge, firms will account for conditions (5.18) and (5.19) in their profit maximization and hence the residual demand against which they maximize their profits is implicitly specified by (5.23), (5.24) and

(5.25). Thus, in determining their profit maximizing output levels the firms try to influence the price at the reference bus and the shadow prices on the common knowledge constraints in the set $L_1 \cup L_2$, while behaving as price takers with respect to the ISO nodal price markup components reflecting congestion on all other transmission lines. Mathematically, each firm g solves the following profit-maximization problem:

$$\max_{\{q_i\}_{i\in N_g},\, p,\, \{\lambda_l^-\}_{l\in L_1},\, \{\lambda_l^+\}_{l\in L_2}} \sum_{i\in N_g}\left(p - \sum_{l\in L_1} D_{li}\lambda_l^- + \sum_{l\in L_2} D_{li}\lambda_l^+ + \tilde{\varphi}_i\right)q_i - \sum_{i\in N_g} C_i(q_i)$$

$$0 \le q_i \le \bar{q}_i, \qquad i \in N_g$$

$$\sum_{i\in N}\left(P_i^{-1}\left(p - \sum_{l\in L_1} D_{li}\lambda_l^- + \sum_{l\in L_2} D_{li}\lambda_l^+ + \tilde{\varphi}_i\right) - q_i\right) = 0$$

$$\bar{k}_l + \sum_{i\in N} D_{li}\left(P_i^{-1}\left(p - \sum_{m\in L_1} D_{mi}\lambda_m^- + \sum_{m\in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i\right) - q_i\right) = 0, \qquad l \in L_1$$

$$\bar{k}_l - \sum_{i\in N} D_{li}\left(P_i^{-1}\left(p - \sum_{m\in L_1} D_{mi}\lambda_m^- + \sum_{m\in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i\right) - q_i\right) = 0, \qquad l \in L_2$$

If we let ρ_i^-, ρ_i^+, α_g, β_{gl}^- and β_{gl}^+ be the Lagrange multipliers corresponding to the constraints, the KKT conditions for this problem are

- with respect to q_i

$$p - \sum_{l\in L_1} D_{li}\lambda_l^- + \sum_{l\in L_2} D_{li}\lambda_l^+ + \tilde{\varphi}_i - \alpha_g + \sum_{l\in L_1} D_{li}\beta_{gl}^- - \sum_{l\in L_2} D_{li}\beta_{gl}^+ - C'(q_i) + \rho_i^- - \rho_i^+ = 0, \quad i \in N_g$$

- with respect to p

$$\alpha_g \sum_{i\in N} \frac{dP_i^{-1}\left(p - \sum_{m\in L_1} D_{mi}\lambda_m^- + \sum_{m\in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i\right)}{dp}$$
$$+ \sum_{l\in L_1}\beta_{gl}^- \sum_{i\in N} D_{li}\, \frac{dP_i^{-1}\left(p - \sum_{m\in L_1} D_{mi}\lambda_m^- + \sum_{m\in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i\right)}{dp}$$
$$- \sum_{l\in L_1}\beta_{gl}^+ \sum_{i\in N} D_{li}\, \frac{dP_i^{-1}\left(p - \sum_{m\in L_1} D_{mi}\lambda_m^- + \sum_{m\in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i\right)}{dp} + \sum_{i\in N_g} q_i = 0$$

- with respect to λ_l^-

$$
\begin{aligned}
&\alpha_g \sum_{i\in N} \frac{dP_i^{-1}\left(p-\sum_{m\in L_1} D_{mi}\lambda_m^- + \sum_{m\in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i\right)}{d\lambda_l^-} \\
&+\sum_{k\in L_1}\beta_{gk}^- \sum_{i\in N} D_{ki}\frac{dP_i^{-1}\left(p-\sum_{m\in L_1} D_{mi}\lambda_m^- + \sum_{m\in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i\right)}{d\lambda_l^-} \\
&-\sum_{k\in L_1}\beta_{gk}^+ \sum_{i\in N} D_{ki}\frac{dP_i^{-1}\left(p-\sum_{m\in L_1} D_{mi}\lambda_m^- + \sum_{m\in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i\right)}{d\lambda_l^-} + \sum_{i\in N_g} D_{li} q_i = 0
\end{aligned}
$$

- with respect to λ_l^+

$$
\begin{aligned}
&\alpha_g \sum_{i\in N} \frac{dP_i^{-1}\left(p-\sum_{m\in L_1} D_{mi}\lambda_m^- + \sum_{m\in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i\right)}{d\lambda_l^+} \\
&+\sum_{k\in L_1}\beta_{gk}^- \sum_{i\in N} D_{ki}\frac{dP_i^{-1}\left(p-\sum_{m\in L_1} D_{mi}\lambda_m^- + \sum_{m\in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i\right)}{d\lambda_l^+} \\
&-\sum_{k\in L_1}\beta_{gk}^+ \sum_{i\in N} D_{ki}\frac{dP_i^{-1}\left(p-\sum_{m\in L_1} D_{mi}\lambda_m^- + \sum_{m\in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i\right)}{d\lambda_l^+} + \sum_{i\in N_g} D_{li} q_i = 0
\end{aligned}
$$

- with respect to ρ_i^-

$$0 \le \rho_i^- \quad \perp \quad q_i \ge 0, \qquad i \in N_g$$

- with respect to ρ_i^+

$$0 \le \rho_i^+ \quad \perp \quad \bar{q}_i - q_i \ge 0, \qquad i \in N_g$$

- with respect to α_g

$$\sum_{i\in N}\left(P_i^{-1}\left(p-\sum_{m\in L_1} D_{mi}\lambda_m^- + \sum_{m\in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i\right) - q_i \right) = 0$$

- with respect to β_{gl}^-

$$\bar{k}_l + \sum_{i\in N} D_{li}\left(P_i^{-1}\left(p-\sum_{m\in L_1} D_{mi}\lambda_m^- + \sum_{m\in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i\right) - q_i \right) = 0, \qquad l \in L_1$$

- with respect to β_{gl}^+

$$\bar{k}_l - \sum_{i\in N} D_{li}\left(P_i^{-1}\left(p-\sum_{m\in L_1} D_{mi}\lambda_m^- + \sum_{m\in L_2} D_{mi}\lambda_m^+ + \tilde{\varphi}_i\right) - q_i \right) = 0, \qquad l \in L_2$$

5.5.2 The Market Equilibrium Conditions

An aggregation of the KKT conditions for the firms' and the ISO' problem leads to the market equilibrium conditions as a non-linear mixed complementarity problem. In the sequel, we assume linear demand and linear marginal cost functions, and present these market equilibrium conditions as a mixed linear complementarity problem (mixed LCP, see [5]):

$$\alpha_i - (r_i + q_i) b_i - p + \sum_{l \in L} \left(\lambda_{li}^- - \lambda_{li}^+ \right) D_{li} = 0, \quad i \in N$$

$$\sum_{i \in N} r_i = 0$$

$$0 \le \lambda_i^- \quad \perp \quad \bar{k}_l + \sum_{i \in N} D_{li} r_i \ge 0, \quad l \notin L_1 \cup L_2$$

$$0 \le \lambda_i^+ \quad \perp \quad \bar{k}_l - \sum_{i \in N} D_{li} r_i \ge 0, \quad l \notin L_1 \cup L_2$$

$$\bar{k}_l + \sum_{i \in N} D_{li} r_i = 0, \quad \lambda_i^+ = 0, \quad l \in L_1$$

$$\bar{k}_l - \sum_{i \in N} D_{li} r_i = 0, \quad \lambda_i^- = 0, \quad l \in L_2$$

$$p - \sum_{l \in L_1} D_{li} \lambda_l^- + \sum_{l \in L_2} D_{li} \lambda_l^+ + \tilde{\varphi}_i - \alpha_g + \sum_{l \in L_1} D_{li} \beta_{gl}^- - \sum_{l \in L_2} D_{li} \beta_{gl}^+ - c_i - d_i q_i + \rho_i^- - \rho_i^+ = 0, \qquad i \in N$$

$$-\alpha_g \sum_{i \in N} \frac{1}{b_i} - \sum_{l \in L_1} \beta_{gl}^- \sum_{i \in N} D_{li} \frac{1}{b_i} + \sum_{l \in L_2} \beta_{gl}^+ \sum_{i \in N} D_{li} \frac{1}{b_i} + \sum_{i \in N_g} q_i = 0,$$

$$\alpha_g \sum_{i \in N} \frac{D_{li}}{b_i} + \sum_{m \in L_1} \beta_{gl}^- \sum_{i \in N} \frac{D_{mi} D_{li}}{b_i} - \sum_{m \in L_2} \beta_{gl}^+ \sum_{i \in N} \frac{D_{mi} D_{li}}{b_i} + \sum_{i \in N_g} D_{li} q_i = 0, \quad l \in L_1$$

$$\alpha_g \sum_{i \in N} \frac{D_{li}}{b_i} - \sum_{m \in L_1} \beta_{gl}^- \sum_{i \in N} \frac{D_{mi} D_{li}}{b_i} + \sum_{m \in L_2} \beta_{gl}^+ \sum_{i \in N} \frac{D_{mi} D_{li}}{b_i} + \sum_{i \in N_g} D_{li} q_i = 0, \quad l \in L_2$$

$$0 \le \rho_i^- \quad \perp \quad q_i \ge 0, \quad i \in N$$

$$0 \le \rho_i^+ \quad \perp \quad \bar{q}_i - q_i \ge 0, \quad i \in N$$

$$\tilde{\varphi}_i = -\sum_{l \in L \setminus L_1 \setminus L_2} (\lambda_l^- - \lambda_l^+) D_{li}$$

For linear demand and linear marginal cost functions, the ISO's and the firm's problems are concave-maximizing problems, and the KKT conditions are necessary and sufficient for characterizing the global optimum of each agent.

5.6 Numerical Example of Equilibrium with Common Knowledge Constraints

This example employs the same six bus network structure as shown in Fig. 5.1 again assuming that all eight lines are identical in terms of their electrical characteristics but line 3-5 has a very low thermal limit of 2MW. The supply and demand data is changed in order to highlight the features of this model. The demand functions are linear with parameter specified in Table 5.7. This system has four generators; the generators at nodes 1 and 2 have a constant marginal cost of 30 $/MWh and the generators at nodes 4 and 6 have a relatively lower constant marginal cost of 10 $/MWh (see Table 5.8). In addition, these four generators are divided into a duopoly structure with the units at nodes 1 and 4 owned by firm #1 and the other two by firm #2.

Table 5. 7: Demand functions

Node	*Demand function*	
	Price intercept ($/MWh)	*Slope*
1	*50*	*1.00*
2	*50*	*0.82*
3	*50*	*1.13*
4	*50*	*1.10*
5	*50*	*0.93*
6	*50*	*0.85*

Table 5.8: Generator information

Node	*Capacity (MW)*	*Owner*	*Marginal cost ($/MWh)*
1	100	Firm #1	30
2	100	Firm #2	30
4	100	Firm #1	10
6	100	Firm #2	10

We compute the market equilibrium of the model with a common knowledge constraint (the congestion on line 3-5), and compare it to that from the pure Bertrand model ignoring this constraint. In this case, the capacities of the generators are not binding in the market equilibrium. We find that, with the common knowledge constraint, both firms increase the output of the generators at nodes 1 and 2 and reduce the production of the units at nodes 4 and 6 (see Table 5.9). As a result, the prices at nodes 1 through 3 are reduced, but the prices at the other three nodes are raised (see Table 5.10). In addition,

Table 5.11 illustrates greater profit for both firms. This suggests that, when recognizing common knowledge constraints, firms behave less competitively which can be explained by the fact that capacitated lines only shift local damand horizontally but do not increase the elasticity of the residual demand functions.

Table 5.9: Generator Output (MW)

Node	*Model*	
	Bertrand	*Common knowledge constraints*
1	0	8.4278
2	0	9.4984
3	0	0
4	79.4605	70.9568
5	0	0
6	48.7073	38.5607

Table 5.10: Prices ($/MWh)

Node	*Model*	
	Bertrand	*Common knowledge constraints*
1	42.3174	37.9236

2	37.4088	34.7477
3	47.2261	41.0994
4	22.6828	25.2201
5	12.8655	18.8683
6	17.7741	22.0442

Table 5.11: Firm's profit ($/h)

	Model	
	Bertrand	*Common knowledge constraints*
Firm #1	1007.7816	1146.7481
Firm #2	378.6554	509.5283
Total	1386.4371	1656.2764

Next, we study how generators' capacities affect the firms' ability to restrict the output from the low cost generators. In doing so, we assume that all capacities are reduced by 50% so that, in the Bertrand model, both low cost generators produce at the full capacity. Unlike in the previous case with nonbinding generation capacities, the equilibrium corresponding to the common knowledge constraint now shows that the units at nodes 1 and 6 reduce their output. Thus, binding generation capacities lead to different behavior of the firms. Indeed, Table 5.14 reports that firm #1's profit decreases.

Table 5.12: Generation with constrained generation capacities (MW)

Node	*Model*	
	Bertrand	*Common knowledge constraints*
1	13.5491	13.2613
2	0	9.0971

4	50.0000	50.0000
6	50.0000	47.0237

Finally, we point out that, due to loop flows in electricity networks, recognizing one particular common knowledge constraint might suffice to relieve congestion on other lines. For example, if the capacity of line 1-2 is 5MW, the pure Bertrand model produces an equilibrium where both lines 1-2 and 3-5 are congested, whereas the equilibrium with the common knowledge constraint indicates that only line 3-5 is capacitated and while the flow on line 1-2 is 4.6MW.

Table 5.13: Prices with constrained generation capacities ($/MWh)

Node	*Model*	
	Bertrand	*Public knowledge constraints*
1	40.1432	37.9236
2	36.9665	35.2421
3	43.3199	40.6050
4	27.4365	27.1978
5	21.0832	21.8349
6	24.2598	24.5163

Table 5.14: Firm's profit ($/h)

	Model	
	Bertrand	*Public knowledge constraints*
Firm #1	1009.2562	964.9672
Firm #2	712.99	730.2980

Total	1722.2462	1695.2653

5.7 Concluding Remarks

This paper examines two approaches for dealing with a limitation of existing nash equilibrium models of congestion prone electricity systems. These models either ignore the effects of joint ownership of generators and the effect of competitive interaction on the elasticity of the residual demand functions faced by the generators, or overestimate the effect of competitive interaction even when transmission capacity is limited or exausted. To address these shortcomings we first develop a hybrid Bertrand-Cournot model of electricity markets with multiple subnetworks. In this model, firms behave *a la* Cournot with respected to the ISO's inter-subnetwork transmission quantities, but *a la* Bertrand with respect to the intra-subnetwork transmission prices. This gives the modeler more flexibility as to how transmission price conjectures are represented in the model compared to pure Cournot or Bertrand models. When affine demand functions and quadratic cost functions are assumed, the market equilibrium conditions of this model become a linear complementarity problem with a bisymmetric positive semi-definite matrix. Numerical examples demonstrate that this model can lead to more realistic market equilibria.

In cases where the network cannot be partitioned into subnetworks as assumed by the hybrid Bertrand-Cournot approach we propose a Bertrand type model where certain systematically congested lines are treated as common knowledge constraints and taken into consideration by the competing firms in assessing their residual demand and optimizing their output levels.

An important limitation of these approaches is that the definition of subnetworks as well as the designation of common knowledge constraints are exogenous. It is possible, for instance, that a Cournot conjecture about flows into a subnetwork is appropriate at some times (when congestion is more likely), but the Bertrand conjecture is preferable at other, less congested periods. Likewise a transmission interface may be congested most of the time but cannot be assumed to be congested all the time. Further research is needed to determine whether it is possible to endogenously determine the appropriate conjecture (perhaps in some iterative fashion inspired by [1]). Empirical research is also desirable to determine what conjectures are actually held by firms in real markets.

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