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Bound Tightening for the Alternating Current Optimal Power Flow Problem

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***Abstract*—We consider the Alternating Current Optimal Power Flow (ACOPF) problem, formulated as a nonconvex Quadratically-Constrained Quadratic Program (QCQP) with complex variables. ACOPF may be solved to global optimality with a semidefinite programming (SDP) relaxation in cases where its QCQP formulation attains zero duality gap. However, when there is positive duality gap, no optimal solution to the SDP relaxation is feasible for ACOPF. One way to find a global optimum is to partition the problem using a spatial branch-and-bound method. Tightening upper and lower variable bounds can improve solution times in spatial branching by potentially reducing the number of partitions needed. We propose special-purpose closed-form bound tightening methods to tighten limits on nodal powers, line flows, phase angle differences, and voltage magnitudes. Computational experiments are conducted using a spatial branch-and-cut solver. We construct variants of IEEE test cases with high duality gaps to demonstrate the effectiveness of the bound tightening procedures.**

***Index Terms*—Optimal power flow, conic optimization, spatial branch and bound, bound tightening.**

NOMENCLATURE

Sets and Indices

N	Set of nodes
n, m	Nodes, $n, m \in N$
$\mathcal{C}(m)$	Set of nodes connected to m

Variables

V_n	Complex voltage
θ_n	Voltage angle
θ_{mn}	Angle between bus m and n : $\theta_m - \theta_n$
P_n, Q_n	Nodal real and reactive power, respectively
P_{mn}, Q_{mn}	Net real and reactive power flow from m to n, respectively

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Parameters

Y	Nodal admittance matrix
G, B	Nodal conductance and susceptance matrix, respectively ($Y := G + jB$)
D_P, D_Q	Vectors of real and reactive load, respectively
$P^{\min}, P^{\max}$	Nodal real power limits
$Q^{\min}, Q^{\max}$	Nodal reactive power limits
$V^{\min}, V^{\max}$	Voltage magnitude limits
$\theta_{mn}^{\min}, \theta_{mn}^{\max}$	Limits on θ_{mn}
c_0, c_1, c_2	Objective coefficients for nodal real power

Operators

$\cdot^*$	Conjugate transpose, i.e., the transpose with entry-wise complex conjugates
$[\cdot]^2$	Element-wise square
$\mathrm{diag}(\,\cdot\,)$	A vector containing diagonal elements of a matrix
$\mathrm{Re}(\,\cdot\,), \mathrm{Im}(\,\cdot\,)$	Real and imaginary components, respectively

I. INTRODUCTION

AN important challenge in power systems optimization is to accurately and tractably model electricity in alternating current (AC) networks. Multi-period problems employ coarser approximations, where transmission losses are eschewed and power flows are linearized so that efficient algorithms can be employed. For certain purposes linearization can perform adequately (see [1]–[3]). However, the inaccuracies that stem from these approximations can result in suboptimal and even infeasible solutions, which may be unacceptable in other cases. It is unclear how much room for improvement may be made by better accounting for AC power flow. However, as Mixed-Integer Programming software for Unit Commitment has shown, even small improvements in operations can have significant overall impact [4].

In this paper we consider a single-period scheduling problem that incorporates steady-state AC power, the Alternating Current Optimal Power Flow (ACOPF). A standard ACOPF problem is to find a minimum cost dispatch of generation and transmission assets to supply load, subject to engineering constraints. Since the AC power flow equations are nonlinear, a common approach to solving ACOPF is through iterative Newton-type solvers (e.g., [5]), which can only guarantee local optimality. Linearization approaches (see [6]) suffer from the same problems. Therefore, in terms of global optimality, the performance of ACOPF methods used in practice remain an open question. The feasible set of ACOPF is nonconvex [7],

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and it is NP-hard to solve [8], [9]. Even finding a feasible solution to ACOPF for radial instances with fixed voltages is NP-hard [10].

Interest in conic optimization techniques for ACOPF is largely due to multiple reports of zero duality gap for various IEEE power system test cases (see [9], [11], [12]). Zero duality gap means that the optimal value of a problem instance coincides with that of the corresponding dual. Throughout the paper we will be referring to the Lagrangian dual of the ACOPF problem formulated as a nonconvex Quadratically-Constrained Quadratic Program (QCQP). The Lagrangian dual of any QCQP can be solved with semidefinite programming (SDP). SDP is a type of conic optimization problem, which is a useful paradigm for global optimization as it guarantees that any local optimum is also globally optimal. Conic optimization can be done with robust methods that can automatically find initial starting points, and have polynomial-time convergence towards the global optimal solution. Even when duality gap exists for ACOPF, by property of weak duality conic relaxations provide a lower bound on the global optimal value, which can be used to judge the quality of candidate feasible solutions. Thus, unlike ACOPF algorithms used in practice, conic optimization can prove a problem is infeasible, or prove that a solution is a global optimum.

For certain network topologies, simple formulations of ACOPF can be solved exactly using a conic relaxation [9], [13], [14]. However, numerous examples demonstrate that zero duality gap cannot be guaranteed in general (e.g., [15]–[18]). In cases with positive duality gap, more advanced techniques must be developed to search for global optimal solutions, such as higher moment relaxations [19], [20]. The method of higher moment relaxations involves solving a sequence of convex problems that grow rapidly in problem size. Several papers have considered an alternative method to global optimization, a spatial branch-and-bound algorithm (see [11], [17], [21]). These algorithms all use lifted relaxations that are computationally expensive to solve for large instances. Therefore practical global optimization for ACOPF remains an active research area.

This paper provides computationally efficient methods for bound tightening, namely the tightening of voltage magnitude, line flow, and phase angle limits. Bound tightening reduces the domain of a problem by removing infeasible regions. This can improve the quality of a relaxation while maintaining its validity; this is true of any relaxation, whether SDP, SOCP, or LP-based (e.g., [22], [23]). For computational experiments we have applied bound tightening to the spatial branch-and-cut algorithm introduced in Chen, Atamtürk, and Oren [24]. Although domain reduction has a natural application to global optimization as it can tighten relaxations, it can be applied elsewhere. For instance, bound tightening could improve the warm-start for an iterative optimization method, or it may complement the low-rank SDP-based heuristic introduced in Sojoudi, Madani and Lavaei [25]. In addition to the bound tightening methods, we also construct modified IEEE test cases with large duality gap that may pose a challenge to global optimization algorithms. Computational tests using a complex QCQP solver (see [24]) show that bound tightening improves the solver's convergence rate on these hard problems. The instances are made publicly available at https://sites.google.com/site/cchenresearch/.

The rest of the paper is organized as follows: Section II describes ACOPF and a SDP relaxation for it; Section III introduces new ACOPF instances with large duality gap; Section IV details the bound tightening procedures; Section V contains computational results; Section VI concludes the paper.

II. Formulations

We present a basic optimal power flow formulation:

$$(\mathbf{ACOPF}): \min c_2^T[P+D_P]^2 + c_1^T(P+D_P) + c_0$$

subject to

$$P + jQ = \operatorname{diag}(Y^* V V^*), \tag{1a}$$
$$P^{\min} \le P + D_P \le P^{\max}, \tag{1b}$$
$$Q^{\min} \le Q + D_Q \le Q^{\max}, \tag{1c}$$
$$[V^{\min}]^2 \le \operatorname{diag}(VV^*) \le [V^{\max}]^2, \tag{1d}$$
$$\tan(\theta_{mn}^{\min})\operatorname{Re}(V_m V_n^*) \le \operatorname{Im}(V_m V_n^*) \le \tan(\theta_{mn}^{\max})\operatorname{Re}(V_m V_n^*). \tag{1e}$$

The power flow equations are modeled with constraint (1a), and nodal and generation power limits with constraints (1b) and (1c). Constraint (1d) enforces voltage magnitude limits, and constraint (1e) enforces bus angle difference limits. Bus angle differences can be recovered with $\theta_{mn} = \arctan(\operatorname{Im}(V_m V_n^*)/\operatorname{Re}(V_m V_n^*))$. Note that for notational brevity we have left out line limits, but three types are explicitly considered in Section IV.

Following Lavaei and Low [9], we consider the following lifted SDP relaxation:

$$(\mathbf{RACOPF}): \min c_2^T[P+D_P]^2 + c_1^T(P+D_P) + c_0$$

subject to

$$P = \operatorname{diag}(GW - BT), \tag{2a}$$
$$Q = \operatorname{diag}(-BW - GT), \tag{2b}$$
$$P^{\min} \le P + D_P \le P^{\max}, \tag{2c}$$
$$Q^{\min} \le Q + D_Q \le P^{\max}, \tag{2d}$$
$$[V^{\min}]^2 \le \operatorname{diag}(W) \le [V^{\max}]^2, \tag{2e}$$
$$\tan\left(\theta_{mn}^{\min}\right) W_{mn} \le T_{mn} \le \tan\left(\theta_{mn}^{\max}\right) W_{mn}, \tag{2f}$$
$$W + jT \succeq 0. \tag{2g}$$

The decision vector V has been replaced by the Hermitian decision matrix $W + jT$, and a rank-one condition on $W + jT$ has been relaxed.

III. New Instances With Large Duality Gap

First we provide some intuition regarding the construction of cases with duality gap. Provided ACOPF is feasible, RACOPF has the same optimal cost if and only if there exists an optimal solution to RACOPF with rank 1. We use an alternative condition from Chen, Atamtürk, and Oren [24]:

Proposition 1: For $n > 1$ a nonzero Hermitian positive semidefinite $n \times n$ matrix X has rank one iff all its 2×2 principal minor determinants are zero.

Proof: Suppose X has rank $r > 1$. Since X is Hermitian it has an $r \times r$ nonzero principal minor. Since X is positive semidefinite this principal minor corresponds to a positive definite $r \times r$ submatrix. As $r \ge 2$, this implies there exists a 2×2

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strictly positive principal minor. Now suppose instead that X has a strictly positive 2×2 principal minor. Then X contains a rank-two principal submatrix and thus $r > 1$. □

Suppose we are given an ACOPF-optimal solution with voltages $\hat{V}$ and powers $\hat{P}, \hat{Q}$. Consider its corresponding rank-one optimal solution in lifted space $\hat{W} + j\hat{T} = \hat{V}\hat{V}^*$. From Proposition 1, we have that the principal minor condition $\hat{W}_{mn}^2 + \hat{T}_{mn}^2 = \hat{W}_{mm}\hat{W}_{nn}$ holds across all bus pairs. The positive semidefinite constraint (2g) enforces $\hat{W}_{mn}^2 + \hat{T}_{mn}^2 \leq \hat{W}_{mm}\hat{W}_{nn}$, so a gap between RACOPF and ACOPF can only occur if for each optimal solution to RACOPF there exists at least one pair m, n such that $W_{mn}^2 + T_{mn}^2 < W_{mm}W_{nn}$.

If there is a pair m, n where decreasing either $\hat{T}_{mn}$ or $\hat{W}_{mn}$ improves the objective value, then there is a gap between the optimal values of RACOPF and ACOPF. A decrease in the magnitude of $\hat{T}_{mn}$ has the equivalent effect of a decrease in the magnitude of θ_{mn}. Note that this is a nonphysical effect if the rank condition is lost; for instance this could lead to an increase in power factor between m and n without affecting flows elsewhere. Decreasing $\hat{W}_{mn}$ decreases real and reactive power at both buses by decreasing real and reactive power flows in both directions across the connecting lines.

Let us now consider conditions where decreasing the magnitudes of $\hat{T}_{mn}$ or $\hat{W}_{mn}$ (and adjusting nodal powers accordingly) could improve the objective function. Since decreasing $|\hat{T}_{mn}|$ reduces the power transfer between m and n, then it may be desirable when line congestion is problematic, or when transfer across a lossy transmission line is otherwise unavoidable. Decreasing $\hat{W}_{mn}$ increases losses, which would allow an otherwise unmanageable amount of power to be produced by dissipating flows in an unphysical manner. Using this intuition, we construct new cases with large duality gap by applying simple changes to IEEE test cases. We name these cases as follows: $9Na, 9Nb, 14S, 14P$, and $118IN$, with number indicating the number of buses and letter indicating the type of change. To the best of our knowledge, the use of negative costs and alternative types of line limits is a novelty in the construction of cases with duality gap. Known instances in the literature use methods including the creation of excess power by changing nodal generation/load limits [17], [18], [20], [21], restricting apparent power flow [15], [20], [21], and changing voltage magnitude limits [18], [20].

Negative Costs $9Na$ and $9Nb$: We use the 9-bus instance in MATPOWER and change the cost coefficients by setting all quadratic coefficients c_2 to be zero, and reversing the sign of the linear real power cost coefficients c_1 on certain generators, making these costs negative. Negative cost coefficients can model opportunity costs such as start-up and shut-down cost avoidance, ramping considerations (e.g., anticipating high demand in the next period), feed-in-tariffs from renewable resources, and the value of absorbing excess generation from an import bus. Thus the cost coefficients in these cases represent bids rather than marginal generation costs. For this small 9-bus network we have constructed extreme cases: in $9NA$ we give negative costs to generators 1 and 3, and in $9NB$ we do so for all three generators.

Congestion: $14S$ and $14P$: We modified the IEEE 14-bus case by applying a a universal line limit across all lines, applying either real power ($14P$) or apparent power ($14S$) limits. For $14P$ we apply a per-unit limit of 0.23, and for 14S a per-unit limit of 0.24. These produce severe amounts of congestion, as further lowering the limit on either case by 0.01 resulted in infeasibility.

Congestion and Negative Costs: $118IN$: We modified the 118-bus IEEE case in order to construct a relatively large case with modest duality gap. The first (in lexicographic order) 7 generators were set to have negative linear costs, and all quadratic cost coefficients were set to zero. Substantial congestion was created by setting a thermal limit across all lines of 1.14 p.u. on current magnitude.

IV. Bound Tightening Procedures for ACOPF

In this section we propose fast procedures for domain reduction aka bound tightening. The typical procedure for bound tightening is as follows: minimize/maximize the desired variable subject to the constraints of the relaxation; however, this is computationally intensive. Instead we consider ACOPF-specific methods with closed-form solutions based on general principles described in Chen, Atamtürk, and Oren [24].

Tightening on Power Flows: Let us examine some bus m. If we consider voltage magnitude and angle constraints at all buses, and real and reactive power constraints only at bus m, then we have the following ACOPF relaxation in polar coordinates:

$$V_n^{\min} \leq |V_n| \leq V_n^{\max}, \tag{3a}$$
$$\theta_{mn}^{\min} \leq \theta_{mn} \leq \theta_{mn}^{\max}, \tag{3b}$$
$$P_m^{\min} \leq G_{mm}|V_m|^2 + \sum_{\forall n \in \mathcal{C}(m)} G_{mn}|V_m||V_n|\cos(\theta_{mn}) + \sum_{\forall n \in \mathcal{C}(m)} B_{mn}|V_m||V_n|\sin(\theta_{mn}) \leq P_m^{\max}, \tag{3c}$$
$$Q_m^{\min} \leq -B_{mm}|V_m|^2 - \sum_{\forall n \in \mathcal{C}(m)} B_{mn}|V_m||V_n|\cos(\theta_{mn}) + \sum_{\forall n \in \mathcal{C}(m)} G_{mn}|V_m||V_n|\sin(\theta_{mn}) \leq Q_m^{\max}. \tag{3d}$$

We can further relax the problem by decoupling P and Q, and rewriting some terms using the optimal solution to certain subproblems. First let us define the following terms:

$$p_{mn} := G_{mn}|V_n|\cos(\theta_{mn}) + B_{mn}|V_n|\sin(\theta_{mn}),$$
$$q_{mn} := -B_{mn}|V_n|\cos(\theta_{mn}) + G_{mn}|V_n|\sin(\theta_{mn}),$$
$$p_m := \sum_{\forall n \in \mathcal{C}(m)} p_{mn},$$
$$q_m := \sum_{\forall n \in \mathcal{C}(m)} q_{mn}.$$

Thus we can rewrite the nodal power equations:

$$P_m = G_{mm}|V_m|^2 + p_m|V_m|,$$
$$Q_m = -B_{mm}|V_m|^2 + q_m|V_m|.$$

We can obtain upper and lower bounds on p_{mn}, q_{mn} by finding the following optima:

$$p_{mn}^U := \max G_{mn}|V_n|\cos(\theta_{mn}) + B_{mn}|V_n|\sin(\theta_{mn})$$
subject to constraints (3a) and (3b),
$$p_{mn}^L := \min G_{mn}|V_n|\cos(\theta_{mn}) + B_{mn}|V_n|\sin(\theta_{mn})$$

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$$
\begin{aligned}
&\text{subject to constraints (3a) and (3b),}\\
&q_{mn}^U := \max -B_{mn}|V_n|\cos(\theta_{mn}) + G_{mn}|V_n|\sin(\theta_{mn})\\
&\text{subject to constraints (3a) and (3b),}\\
&q_{mn}^L := \min -B_{mn}|V_n|\cos(\theta_{mn}) + G_{mn}|V_n|\sin(\theta_{mn})\\
&\text{subject to constraints (3a) and (3b),}\\
&p_m^L := \sum_{\forall n\in\mathcal{C}(m)} p_{mn}^L,\\
&p_m^U := \sum_{\forall n\in\mathcal{C}(m)} p_{mn}^U,\\
&q_m^L := \sum_{\forall n\in\mathcal{C}(m)} q_{mn}^L,\\
&q_m^U := \sum_{\forall n\in\mathcal{C}(m)} q_{mn}^U.
\end{aligned}
$$

Each bound is computed by checking variable bounds and the unconstrained first-order necessary conditions (FONC). For instance, for p_{mn}, FONC give us either $\theta_{mn}^* = \arctan((B_{mn})/(G_{mn}))$ if $G_{mn} \neq 0$, or else $\theta_{mn}^* = (\pi)/(2)$, in which case we can discard the point as infeasible. We can test all candidates ($\theta_{mn} = \arctan((B_{mn})/(G_{mn})), |V_n| = |V_n|^{\min}; \theta_{mn} = \theta_{mn}^{\max}, |V_n| = |V_n|^{\max}$, etc.) to determine bounds for p_m, q_m. With these variable bounds we form the following relaxation:

$$
\begin{aligned}
&V_m^{\min} \le |V_m| \le V_m^{\max},\\
&P_m^{\min} \le G_{mm}|V_m|^2 + p_m|V_m| \le P_m^{\max},\\
&Q_m^{\min} \le -B_{mm}|V_m|^2 + q_m|V_m| \le Q_m^{\max},\\
&p_m^L \le p_m \le p_m^U,\\
&q_m^L \le q_m \le q_m^U.
\end{aligned}
$$

From here, new variable bounds are determined in a straightforward and computationally efficient manner. Let us consider the power bounds first. An over/underestimate of the max/min real/reactive power flow is attained either at variable bounds or the appropriate unconstrained FONC point. For P we have:

$$
\begin{aligned}
&\min\{P^N, P^A\} \le G_{mm}|V_m|^2 + p_m|V_m| \le \max\{P^X, P^B\},\\
&P^N := \min\Big\{G_{mm}\left(V_m^{\min}\right)^2 + p_m^L\left(V_m^{\min}\right),\\
&G_{mm}\left(V_m^{\max}\right)^2 + p_m^L\left(V_m^{\max}\right)\Big\},\\
&P^X := \max\Big\{G_{mm}\left(V_m^{\min}\right)^2 + p_m^U\left(V_m^{\min}\right),\\
&G_{mm}\left(V_m^{\max}\right)^2 + p_m^U\left(V_m^{\max}\right)\Big\},\\
&V_P^A := \begin{cases} -\frac{p_m^L}{2G_{mm}}, & G_{mm}\neq 0, V_m^{\min} \le -\frac{p_m^L}{2G_{mm}} \le V_m^{\max},\\ V_m^{\min} & o/w,\end{cases}\\
&V_P^B := \begin{cases} -\frac{p_m^U}{2G_{mm}}, & G_{mm}\neq 0, V_m^{\min} \le -\frac{p_m^U}{2G_{mm}} \le V_m^{\max},\\ V_m^{\max} & o/w,\end{cases}\\
&P^A := G_{mm}\left(V_P^A\right)^2 + p_m^L\left(V_P^A\right),\\
&P^B := G_{mm}\left(V_P^B\right)^2 + p_m^U\left(V_P^B\right).
\end{aligned}
$$

Here, V_P^A, V_P^B are FONC solutions for V_m given a fixed value of p_m, and P^A, P^B are the corresponding nodal powers. Reactive power can be updated in the same way, replacing G_{mm} with $-B_{mm}$, and p with q. Thus voltage limits can be used to tighten nodal power limits.

We can apply the quadratic root formula to make inferences about voltage magnitude using real and reactive power constraints. Note that only the positive root needs to be considered, as the negative part corresponds to the lower portion of the nose curve that is avoided in power systems operation to maintain stability. Let us consider the following problem structure:

$$
\begin{aligned}
&ax^2 + bx + c \le 0,\\
&b^L \le b \le b^U,\\
&\equiv\\
&ax^2 + bx + c + d = 0,\\
&d \ge 0,\\
&b^L \le b \le b^U.
\end{aligned}
$$

Here, a, c are parameters, and x, b, d are real-valued variables. For example, with the real power upper bound we have $a = G_{mm}, b = p_m, c = -P^{\max}, x = |V_m|$, and hence tightening of the limits on voltage magnitude. We are interested in the projection on x so that we can establish implied variable bounds for purposes of tightening: $x^L \le x \le x^U$. We use the property that the lower portion of nose curves are forbidden or infeasible regions and take only the higher root:

$$
x = \frac{\sqrt{b^2 - 4ac - 4ad}}{2|a|} - \frac{b}{2a}.
$$

From here, by maximizing or minimizing the right-hand side with b, d, we infer either upper and lower bounds x^L, x^U, or else infeasibility:

$$
\begin{aligned}
&a = 0, b^U < 0:\\
&\quad -\frac{c}{b^L} \le x,\\
&a = 0, b^L > 0, c \le 0:\\
&x \le -\frac{c}{b^L},\\
&a = 0, b^L > 0, c > 0:\\
&\text{Infeasibility,}\\
&a < 0:\\
&\min_{b\in\{b^L,b^U\}} \frac{\sqrt{[b^2-4ac]^+} + b}{-2a} \le x,\\
&a > 0, 4ac \le (b^N)^2:\\
&-\frac{b^U}{2a} \le x \le \max_{b\in\{b^L,b^U\}} \frac{\sqrt{(b)^2 - 4ac} - b}{2a},\\
&a > 0, (b^N)^2 < 4ac \le (b^X)^2, b^X < 0:\\
&\sqrt{\frac{c}{a}} \le x \le \frac{\sqrt{(b^L)^2 - 4ac} - b^L}{2a},\\
&a > 0, (b^N)^2 < 4ac \le (b^X)^2, b^X > 0:\\
&\quad -\frac{b^U}{2a} \le x \le \frac{\sqrt{(b^U)^2 - 4ac} - b^U}{2a},\\
&a > 0, 4ac > (b^X)^2:\\
&\text{Infeasibility,}\\
&b^N := \begin{cases} b^L, & (b^L)^2 < (b^U)^2,\\ b^U & o/w,\end{cases}
\end{aligned}
$$

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$$b^X := \begin{cases} b^L, & (b^L)^2 > (b^U)^2, \\ b^U & o/w. \end{cases}$$

Tightening on Line Constraints: Three types of line flow limits typically used for ACOPF are apparent power, real power, and line current magnitude. For notational simplicity we will assume that nodes are connected by a single line. This is not a technical requirement (we include limits on specific lines in our experiments) as multiple lines can be easily accommodated by replacing m, n entries with the appropriate indices for specific lines.

Apparent power is usually applied as a proxy for thermal line or transformer limits: $P_{mn}^2 + Q_{mn}^2 \le (S_{mn}^{\max})^2$

Note that line limits are quartic constraints with respect to voltages. In our relaxation we include nodal powers as explicit decision variables, so the line limits are modeled as convex quadratic constraints with respect to power, which maintains the QCQP framework.

Let us now deduce some limits by fixing either P_{mn} or Q_{mn} in the same way as with nodal powers. For brevity we will only show the procedure for real power, with the procedure holding symmetrically for reactive power. First let us determine the minimum possible magnitude of reactive power flow:

$$\begin{aligned}
&Q_{mn}^0 := \max\left\{Q_{mn}^N, \min\left\{Q_{mn}^X, 0\right\}\right\}, \\
&Q_{mn}^N := \min\Big\{-B_{mn}\left(V_m^{\min}\right)^2 - q_{mn}^U V_m^{\min}, \\
&\quad - B_{mn}\left(V_m^{\max}\right)^2 - q_{mn}^U V_m^{\max}, \\
&\quad -B_{mn}\left(V_Q^A\right)^2 - q_{mn}^U V_Q^A\Big\}, \\
&Q_{mn}^X := \max\Big\{-B_{mn}\left(V_m^{\min}\right)^2 - q_{mn}^L V_m^{\min}, \\
&\quad - B_{mn}\left(V_m^{\max}\right)^2 - q_{mn}^L V_m^{\max}, \\
&\quad -B_{mn}\left(V_Q^B\right)^2 - q_{mn}^L V_Q^B\Big\}, \\
&V_Q^A := \begin{cases} -\frac{q_{mn}^U}{2B_{mn}}, & B_{mn} \ne 0, V_m^{\min} \le -\frac{q_{mn}^U}{2B_{mn}} \le V_m^{\max}, \\ V_m^{\min} & o/w, \end{cases} \\
&V_Q^B := \begin{cases} -\frac{q_{mn}^L}{2B_{mn}}, & B_{mn} \ne 0, V_m^{\min} \le -\frac{q_{mn}^L}{2B_{mn}} \le V_m^{\max}, \\ V_m^{\max} & o/w. \end{cases}
\end{aligned}$$

Using the same principles as with nodal powers, we have determined Q_{mn}^N, Q_{mn}^X, respectively upper and lower bounds on reactive power flow from m to n. $|Q_{mn}^0|$ gives us the minimum magnitude, so we the have the following valid inequality:

$$\begin{aligned}
P_{mn}^2 + \left(Q_{mn}^0\right)^2 &\le \left(S_{mn}^{\max}\right)^2, \\
&\equiv P_{mn}^{\min} \le P_{mn} \le P_{mn}^{\max}, \\
P_{mn}^{\min} &:= -\sqrt{\left(S_{mn}^{\max}\right)^2 - \left(Q_{mn}^0\right)^2}, \\
P_{mn}^{\max} &:= \sqrt{\left(S_{mn}^{\max}\right)^2 - \left(Q_{mn}^0\right)^2}.
\end{aligned}$$

Note that the explicit real power limits are sometimes included as a proxy for voltage stability limits. In such a case we use whichever bound is tighter. Voltage magnitude bound tightening can then be applied to line flow limits using the same procedure as for nodal power limits. For instance, with $P_{mn}^{\max}$ and we have $G_{mn}|V_m|^2 - p_{mn}|V_m| - P_{mn}^{\max} \le 0$.

Line current magnitude is the key factor in thermal line limit violation. In the lifted space we write this as a linear constraint, unlike apparent power bounds:

$$\begin{aligned}
&|I_{mn}| \le I_{mn}^{\max}, \\
&\quad \equiv \left(G_{mn}^2 + B_{mn}^2\right)\left(|V_m|^2 + |V_n|^2\right. \\
&\qquad \left. - 2|V_m||V_n|\cos(\theta_{mn})\right) \le \left(I_{mn}^{\max}\right)^2, \\
&\quad \implies \left(G_{mn}^2 + B_{mn}^2\right)\left(W_{mm} + W_{nn},\right. \\
&\qquad \left. - 2W_{mn}\right) \le \left(I_{mn}^{\max}\right)^2.
\end{aligned}$$

Defining $\theta_{mn}^0 := \max\{\theta_{mn}^{\min}, \min\{\theta_{mn}^{\max}, 0\}\}$ as the feasible angle closest to zero, we make the following inference, supposing that $|Y_{mn}| \ne 0$:

$$\begin{aligned}
&|V_m| \le \max\{V^D, V^E, V^F\}, \\
&V^C := \begin{cases} \sqrt{\frac{I^R}{1-\cos^2(\theta_{mn}^0)}}, & \theta_{mn}^0 \ne 0, \\ \infty, & o/w, \end{cases} \\
&V^D := \begin{cases} V_n^{\max}\cos\left(\theta_{mn}^0\right) & \\ +\sqrt{I^R - \left(V_n^{\max}\sin\left(\theta_{mn}^0\right)\right)^2}, & V_n^{\max} \le V^C, \\ V^C, & o/w, \end{cases} \\
&V^E := \begin{cases} V_n^{\min}\cos\left(\theta_{mn}^0\right) & \\ +\sqrt{I^R - \left(V_n^{\min}\sin\left(\theta_{mn}^0\right)\right)^2}, & V_n^{\min} \le V^C, \\ \text{Infeasible problem}, & o/w, \end{cases} \\
&V^F := \min\left\{V_n^{\min}, \max\left\{\min\left\{V_n^{\max}, V^C\right\}, V^G\right\}\right\}, \\
&V^G := \begin{cases} \sqrt{I^R}\left(\left|\sin(\theta_{mn}^0)\right| + \right. & \\ \left.\cos\left(\theta_{mn}^0\right)\left|\cot\left(\theta_{mn}^0\right)\right|\right), & \theta_{mn}^0 \ne 0, \\ V^E, & o/w, \end{cases} \\
&I^R := \frac{\left(I_{mn}^{\max}\right)^2}{G_{mn}^2 + B_{mn}^2}.
\end{aligned}$$

The same procedure applies symmetrically on $|V_n|$. We also update the angle bounds, considering only the nontrivial cases where $|Y_{mn}|V_m^{\min}V_n^{\min} \ne 0$:

$$\begin{aligned}
&\left(G_{mn}^2 + B_{mn}^2\right)\left(|V_m|^2 + |V_n|^2,\right. \\
&\quad \left. - 2|V_m||V_n|\cos(\theta_{mn})\right) \le \left(I_{mn}^{\max}\right)^2, \\
&\quad \implies \frac{|V_m|^2 + |V_n|^2 - I^R}{2|V_m||V_n|} \le \cos(\theta_{mn}). \qquad (4)
\end{aligned}$$

We can then determine the minimum of the left-hand side of inequality (4). Supposing a nontrivial bound, where $(V_m^{\min})^2 + (V_n^{\min})^2 - I^R > 0, V_m^{\min} > 0, V_n^{\min} > 0$, by examining derivatives we can see that the minimum is attained at one of the following values:

$$\begin{aligned}
&|V_m| = V_m^{\min}, \\
&|V_n| = \max\left\{V_n^{\min}, \min\left\{V_n^{\max}, \left(V_m^{\min}\right)^2 - I^R\right\}\right\}; \\
&|V_m| = V_m^{\max}, \\
&|V_n| = \max\left\{V_n^{\min}, \min\left\{V_n^{\max}, \left(V_m^{\max}\right)^2 - I^R\right\}\right\}; \\
&|V_n| = V_n^{\min}, \\
&|V_m| = \max\left\{V_m^{\min}, \min\left\{V_m^{\max}, \left(V_n^{\min}\right)^2 - I^R\right\}\right\}; \\
&|V_n| = V_n^{\max}, \\
&|V_m| = \max\left\{V_m^{\min}, \min\left\{V_m^{\max}, \left(V_n^{\max}\right)^2 - I^R\right\}\right\}.
\end{aligned}$$

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TABLE I
COMPARISON WITH AND WITHOUT BOUND TIGHTENING

Case	dgap	With Bound Tightening						Without Bound Tightening					
		Nodes	Depth	Time	rgap	egap	cgap	Nodes	Depth	Time	rgap	egap	cgap
g9	0.28%	57	22	20	0.36%	0.08%	76%	77	27	28	0.36%	0.10%	73%
g14	0.06%	49	18	18	0.16%	0.10%	38%	67	21	28	0.16%	0.10%	40%
g30	0.09%	73	12	45	0.18%	0.09%	49%	75	12	51	0.18%	0.09%	49%
g57	2.22%	109	39	358	2.31%	0.09%	96%	215	68	1202	2.31%	0.10%	96%
m39Q	0.57%	1365	83	2546	3.57%	3.00%	16%	4149	101	5423	3.57%	3.52%	1%
m57L	0.06%	6	5	15	0.16%	0.10%	39%	13	7	23	0.16%	0.10%	38%
m118L	0.66%	128	38	3815	0.76%	0.10%	87%	383	51	5648	0.76%	0.52%	31%
9Na	17.00%	1171	45	340	18.00%	1.00%	94%	2771	70	587	18.00%	1.00%	94%
9Nb	18.29%	1149	43	333	19.29%	1.00%	95%	2325	69	522	19.29%	1.00%	95%
14P	4.32%	2290	50	1026	5.32%	1.00%	81%	4999	72	1766	5.32%	1.00%	81%
14S	1.97%	1799	51	968	2.97%	1.00%	66%	3979	72	1628	2.97%	1.00%	66%
118IN	1.06%	190	33	1620	1.61%	1.00%	38%	241	39	1821	2.05%	0.99%	52%
avg	**3.83%**	**699**	**37**	**925**	**4.56%**	**0.71%**	**65%**	**1608**	**51**	**1561**	**4.59%**	**0.79%**	**60%**

Let θ^A_{mn} be the minimum value for the left-hand side of inequality (4). If $\theta^A_{mn} > 1$, then the problem is infeasible; otherwise, θ_{mn} is bounded above and below by $\pm \arccos(\theta^A_{mn})$.

Tightening on Graph Cycles: For meshed networks, we propagate angle bound changes across cycles using the identity that the sum of angle differences around a cycle must sum to zero. For instance, if we choose to partition the bounds of θ_{mn}, it is easy to check if there is some third bus k that connects to m and n in the chordally completed graph. Supposing that all bounds are at -30 to 30 degrees, and that the upper bound $\hat{\theta}^{\max}_{mn}$ has been updated to -15 degrees, then we can update the other lower bounds:

$$\hat{\theta}^{\min}_{nk} = \max\left\{\theta^{\min}_{nk}, -\left(\hat{\theta}^{\max}_{mn} + \theta^{\max}_{km}\right)\right\} = -15 \text{ deg},$$
$$\hat{\theta}^{\min}_{km} = \max\left\{\theta^{\min}_{km}, -\left(\hat{\theta}^{\max}_{mn} + \theta^{\max}_{nk}\right)\right\} = -15 \text{ deg}.$$

Although this applies to cycles of any size, for simplicity we restrict the procedure to 3-cycles in our experiments. Note that this procedure generalizes to QCQP with bounded complex variables (see [24]).

V. COMPUTATIONAL EXPERIMENTS

A. Setup

All experiments herein are performed with a 2.26 dual-core Intel i3-350M processor and 4 GB main memory. Algorithms are coded in MATLAB (see [26]) with model processing performed by YALMIP (see [27]). We use the solver for QCQP with bounded complex variables developed in Chen, Atamtürk, and Oren [24]. A brief description of the solver follows.

B. Solver

The solver of Chen, Atamtürk, and Oren [24] is designed for general QCQP problems with bounded real or complex variables. A spatial branch-and-cut approach is used, and live nodes are selected using depth-first search strategy. At every live node, the upper bound problem is solved using IPOPT [28]. The lower bound problem is a sparse lifted SDP relaxation that is strengthened with cuts and is solved with MOSEK [29]. A live node is pruned if the lower bound is infeasible or if the lower bound exceeds or else is within a user-specified percentage of the best-known upper bound. If a live node is not pruned, then it is branched upon: an entry of the lifted matrix of the SDP relaxation is selected for branching and its bound is bisected, forming two child nodes. For ACOPF the interpretation is that spatial branching is performed on either a node's voltage magnitude or a voltage phase angle difference between two nodes. In this paper we have used the MVWB branching strategy, which selects nodes based on minimizing a worst-case estimate of rank violation in the relaxation's decision matrix. Bound tightening is applied at every live node in order to tighten the current lower bound and subsequent child node lower bounds, if any. The solver terminates when any one of the following criterion are met: a search tree limit of 10000 nodes explored; a time limit of 1.5 hours; or a user-specified optimality gap criterion is reached (this differs by case and is specified in Section V). Furthermore, all nodes beyond a search tree depth of 100 are pruned. Upon termination, the solver returns the lowest-cost feasible solution found as well as a lower bound on the optimum cost derived from the branch-and-cut procedure.

C. Instances

In addition to the challenging instances produced in Section III, we have included test instances with small duality gap from Gopalakrishnan *et al.* [21]. These instances are called $g9$-$g57$, with the number indicating the number of buses in the problem. Additionally, we have included all cases from Molzahn and Hiskens [20] with duality gap greater than 0.1%. These are named $m39Q, m57L$, and $m118L$, using their naming convention. Since the IEEE test cases do not include phase angle difference limits, we have applied a 30 degree bound for all connected bus pairs. For the challenging problems we set a global optimality tolerance of 1%, and for $g9$-$g57$ we use a tolerance of 0.1%. The solver had trouble converging on case $m39Q$, so we set a higher optimality tolerance of 3%.

D. Results

We summarize our results in Table I. The columns are defined as follows. *Case* is the case name. *dgap* is a lower bound on the duality gap with respect to the standard relaxation; it is established by the best known lower bound calculated by the solver. *Nodes* are the number of search tree nodes explored before termination. *Depth* is the maximum search tree depth. Time is the total time spent in the solver. *rgap* is the root gap, calculated as $(gub - rlb)/|gub|$, where gub is the best known upper bound and rlb is the root node lower bound. *egap* is the end gap, calculated as $(gub - glb)/|gub|$, where glb is the global lower bound established by the solver at termination. *cgap* is the closed gap,

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TABLE II
BREAKDOWN OF TIME SPENT (SECONDS)

Case	With Bound Tightening						Without Bound Tightening					
	Total	LB	UB	OH	Total/n	OH/n	Total	LB	UB	OH	Total/n	OH/n
g9	20	3	11	6	0.35	0.11	28	5	16	7	0.36	0.09
g14	18	5	8	4	0.37	0.09	28	9	14	5	0.42	0.08
g30	45	18	17	10	0.62	0.14	51	20	22	9	0.68	0.12
g57	358	114	190	54	3.28	0.49	1202	233	911	59	5.59	0.27
m39Q	2546	500	1377	669	1.87	0.49	5423	1537	3074	813	1.31	0.20
m57L	15	6	5	3	2.50	0.57	23	13	8	2	1.77	0.23
m118L	3815	818	2604	393	29.81	3.07	5648	1519	3737	392	14.75	1.02
9Na	340	58	173	109	0.29	0.09	587	134	307	147	0.21	0.05
9Nb	333	57	168	107	0.29	0.09	522	117	274	131	0.22	0.06
14P	1026	316	420	290	0.45	0.13	1766	648	811	307	0.35	0.06
14S	968	263	436	269	0.54	0.15	1628	548	804	275	0.41	0.07
118IN	1620	737	553	330	8.52	1.74	1821	878	638	304	7.56	1.26
avg	**925**	**241**	**497**	**187**	**4.07**	**0.60**	**1561**	**472**	**885**	**204**	**2.80**	**0.29**

i.e., $1-(egap)/(rgap)$. Note that $dgap$=$egap$-$rgap$. Column averages are provided in the last row.

For cases with small root gap, $g9$-$g30$ and $m57L$, bound tightening has a modest effect, and the solver terminates quickly regardless. For the more difficult problems bound tightening presents clear advantages in both time and search tree size. Without bound tightening, the solver reached the time limit on cases $m39Q$ and $m118L$ before reaching the optimality criterion. In case $118IN$ bound tightening reduces the root gap by 16%, and for other cases it does not have substantial effect at the root node.

Table II provides a detailed breakdown of times. $Total$ indicates the total time spent in the solver. LB is the time spent solving lower bound problems, UB is the time spent upper bound problems. The overhead (OH) is calculated as $Total - LB - UB$. $Total/n$ is the time spent per search tree node, and OH/n is the overhead time per node. There are small but significant increases in overhead due to the bound tightening procedures. The per-node overhead increase is larger on the more difficult cases. This is because the bound tightening procedure is able to prune a high percentage of nodes due to infeasibility, and thus makes up a larger percentage of total time spent. With the exception of phase angle tightening the procedures presented in this paper involve closed-form solutions and involves only simple arithmetic operations. The angle tightening involves propagation across cycles on chordal graphs (see [24]), which has linear time worst case complexity (see [30]). Thus it may be possible to further reduce per-node time impacts with more efficient coding practices than used in our prototype.

VI. CONCLUSION

We constructed new instances of ACOPF with high duality gap, which require methods beyond a one-stage SDP relaxation for global optimization. We presented closed-form bound-tightening procedures to reduce the domain of the problem by removing infeasible regions. Computational experiments using a spatial branch-and-cut solver indicate that the bound-tightening techniques are particularly effective on more difficult instances.

Future work could consider hybrid bound reduction techniques. For instance, bound tightening using full relaxations could be judiciously applied to problematic buses, and our closed-form methods could then be used to propagate bound changes on neighboring buses. Moreover, further investigation is needed to see if propagating angle bound changes on larger cycles may be useful. It may also be worthwhile to consider bound tightening on other problems involving AC power flow equations, such as optimal capacitor location (see [31]).

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