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Single and Multisettlement Approaches to Market Clearing Under Demand Uncertainty

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Abstract. Participants in electricity markets face a substantial amount of uncertainty, and with increased penetration of volatile renewable generation this uncertainty has further increased. Conventionally designed electricity markets cope with uncertainty by running two markets: a market that is cleared ahead of time, followed by a real-time balancing market to reconcile actual realizations of demand and available generation. In such mechanisms, the initial clearing process does not take into account the distribution of outcomes in the balancing market. Recently, an alternative so-called stochastic settlement market has been proposed (see, e.g., Pritchard et al. 2010 [Pritchard G, Zakeri G, Philpott A (2010) A single-settlement, energy-only electric power market for unpredictable and intermittent participants. *Oper. Res.* 58(4):1210–1219.] and Bouffard et al. 2005 [Bouffard F, Galiana FD, Conejo AJ (2005) Market-clearing with stochastic security—Part I: Formulation. *IEEE Trans. Power Systems* 20(4):1818–1826.]) where the ISO clears both stages in one single settlement market.

While the effectiveness of the stochastic market clearing mechanism is clear when the market is competitive, this is open to question for imperfectly competitive markets. In this paper we consider simplified models for two types of market clearing mechanisms. First, a market clearing mechanism utilized in New Zealand, whereby firms offer in advance and are notified of a clearing quantity and price guide based on an estimate of demand, followed by real-time dispatch. We refer to this as NZTS. Secondly we consider a simplified stochastic programming market clearing mechanism. We compute Nash equilibria of games resulting from each of the market clearing mechanisms. We prove that under the assumption of symmetry, our simplified stochastic programming market clearing is equivalent to a two-period single settlement system that takes account of deviation penalties in the second stage. These, however, differ from NZTS. We show that when we assume symmetry, this stochastic settlement model results in better social welfare than does NZTS. We also investigate a number of asymmetric examples numerically.

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Keywords: uncertainty • stochastic programming • electricity markets • equilibrium • market clearing

1. Introduction

Electricity markets face two key features that set them apart from other markets. The first is that electricity cannot be stored, so demand must equal supply at all times. This necessitates the presence of a real-time (or balancing) market, although it is also imperative that a day-ahead market clears to cater for the participation of slower response units.

Second, electricity is transported from suppliers to load over a transmission network with possible constraints. The combination of these two features means that in almost all electricity markets today, an independent system operator (ISO) sets dispatch centrally and clears the market. Generators and demand-side users can make offers and bids, and the ISO will choose which are accepted according to a predetermined settlement system.

The conventional arrangement of a day-ahead followed by real-time market clearing takes on different nuances in various jurisdictions. One approach used is to run a two-period market clearing model (see, e.g., Kamat and Oren 2004). In many jurisdictions such as the PJM, this amounts to a day-ahead followed by a real-time market with separate, financially binding settlements for each market (see Ott 2003, Zheng and Litvinov 2006). Here, generating firms bid in supply function offers of generation and the utilities (or load serving entities) bids demand functions in the day-ahead market. The market is cleared without participation from volatile renewable generation such as wind. Then in real time firms compete for any deviations introduced in the spot through wind generation, with possibly updated supply and demand functions. We note that almost all such markets are strictly monitored to ensure participants reflect their true cost of generation.

In New Zealand, however, the nature of the market is somewhat different. The New Zealand electricity market (NZEM) is a hydro-dominated market. As it is difficult to obtain an accurate value of water, it is not possible to monitor the NZEM to ensure that generators bid at cost. In the NZEM, generators can place offers for a given half hour period up to "gate closure," which occurs two hours prior to the designated period. At gate closure these offers are locked in. Estimates of dispatch quantity and price are provided to the industry participants based on forecast demand in the periods leading up to real time. We refer to these as predispatch quantities and prices.[1]

At the start of the designated period an accurate measure of demand is available to the ISO and the generators are dispatched accordingly. In the NZEM, the ISO redispatches the generators every five minutes during a half hour period, using updated demand information but according to the same offer curves, locked in at gate closure.[2] In the NZEM there is only a single settlement, as the predispatch quantities and prices are not financially binding. In this two-period single-settlement (NZTS) market, expected demand is used to clear the predispatch quantities and the ISO has no explicit measure of any deviation costs for a generator. Here, the nature of the supply function (different quantities offered at different prices) is relied upon to allow firms to cope with different dispatch levels. While this may have been adequate in systems with limited uncertainty, growing penetration of renewables can lead to large deviations in dispatch at gate closure versus real time. These short-term changes will have associated costs that are not built into a supply function, which operates on the principle that demand may vary but the firm will have more advance warning of this (for an account of these deviation costs we refer the reader to Myles and Herron 2012 and Kumar et al. 2012). We demonstrate in this paper that it is important to reflect these deviation costs in terms of social welfare. This is relevant to some recent debates on complexity versus simplification in electricity markets (see Bushnell 2013).

An alternative to the two-period settlement systems is to use a stochastic settlement process to deal with variable demand. (We will consider demand as gross demand net intermittent renewable generation such as wind, which in the NZEM is forced to offer at zero price.) In a stochastic settlement, the ISO can choose both predispatch and short-run deviations for each generator to maximize expected social welfare in one step. We might then expect a stochastic settlement system to do better (on average) than a two-period system. The idea of a stochastic settlement can be attributed to Bouffard et al. (2005), Wong and Fuller (2007), and Pritchard et al. (2010) and is further analyzed in Morales et al. (2012, 2014a, b), and Zavala et al. (2015). In these two-stage, single settlement models, the predispatch clears with information about the future distribution of uncertainties in the system (e.g., demand and volatile renewable generation) and information about deviation costs for each generator. These models assume that each firms' offers and deviation costs are truthful. In an imperfectly competitive market (where suppliers may offer generation above cost) this assumption is not valid. Can the stochastic settlement auction give better expected social welfare when firms are behaving strategically? This is the question explored in this paper.

To answer this question, we should first construct an equilibrium model of these two market mechanisms. There are many studies that develop equilibrium models for energy and other markets. These include the seminal work of Klemperer and Meyer (1989) that analyzes supply function equilibrium models, as well as various Cournot and linear supply function equilibrium models such as those developed in Downward et al. (2010), Green (1999), Baldick et al. (2004), and Day et al. (2002).

These studies focus on a market with a single settlement. For our investigations, it is necessary to build equilibria over markets with a predispatch followed by balancing clearing, such as the NZTS. Studies that address equilibria obtained on a forward (contract) market followed by a wholesale market yield models that in their two-stage nature are relevant to our analysis. These include Allaz (1992), Allaz and Vila (1993), Willems (2005), and supply function equilibrium models with a preceding contract stage such as Newbery (1998), Anderson and Xu (2005), Gans et al. (1998), as well as Haskel and Powell (1994), Willems (2005), Bushnell (2007), and Su (2007). Our focus for NZTS however is to address the effect of costs (to the generators) due to deviations from dispatch point, resulting from short-term (a few trading periods) variations in net demand. This has not been studied in any of the previous papers.

There are also a number of models in the literature that consider uncertainty in conjunction with forward markets. von der Fehr and Harbord (1992) explained how spot prices could be reduced as an effect of contract forward markets. Shanbhag (2006) investigated a two settlement stochastic market for a two-node network using a Nash-Cournot equilibria. Zhang et al. (2010) proposed a two-stage oligopoly stochastic Nash-Cournot equilibrium problem with equilibrium constraints. None of these models however cater for the recently proposed stochastic programming market clearing mechanism that we investigate in this paper.

Our Contribution

In this paper, we investigate the effects of using a stochastic market clearing mechanism that considers deviation costs and uncertainty instead of the existing deterministic mechanisms (such as New Zealand electricity market). In particular, we define three models, which we analyze as games. We derive the Nash equilibria of these models, and use them to compare the performance of these models:

1. A two-period settlement model (in particular, a simplified version of the New Zealand electricity market) representing a deterministic market clearing mechanism.
2. A single settlement stochastic programming mechanism with deviation costs as part of the offer submitted by the generating firms.
3. A single settlement stochastic programming mechanism with constant deviation penalties.

We start by introducing a simplified version of the NZTS market currently operated in New Zealand. We will then introduce a simplified version of the stochastic programming mechanism for clearing electricity markets. The first simplification in our models is to use affine supply function offers following the work of Green (1996, 1999). He used this restricted form of supply function to assess the effects of policies on enhancing competition in the England and Wales market. He showed that in presence of linear demand functions, it is always possible to find a supply function equilibrium that is an equilibrium over affine supply functions. The linear nature of these supply functions assists us in obtaining a tractable analytical solution. Subsequently, Baldick et al. (2004) extended Green's work to piecewise linear supply functions. Day et al. have also used linear supply functions but with specific forms of conjectural variation (see Day et al. 2002).

In what follows, we will present existence results for equilibria for the simplified NZTS and derive an analytical expression for a symmetric equilibrium (with identical firms). We then establish the key result that reduces our simplified stochastic market clearing mechanism (ISOSP) to a NZTS type model, but with explicit deviation penalties. Here again we construct analytical expressions for symmetric equilibria, for each level of deviation penalty. Finally we compare the symmetric equilibria of NZTS and a special case of ISOSP settlements and show that the ISOSP settlement with fixed, explicit deviation costs performs better in terms of expected social welfare for a large range of deviation penalties that includes the true deviation costs common to the identical firms. When the assumption of symmetry is relaxed, analytical expressions for an equilibrium become intractable for either model. We will therefore present numerical results for a number of cases and compare equilibria of the two market clearing systems. We find that in all cases ISOSP performs better than the NZTS.

In Section 6.2, we repeat the symmetric experiments that construct and compare the equilibria of NZTS and ISOSP mechanisms, but for the restricted version of linear supply functions where the intercept is set to zero. Here we find that there is a unique symmetric equilibrium for the game. For this case, we present an example where the ISOSP welfare is in fact less than the NZTS mechanism. This result is in contrast to the previous set of results where generators have another degree of freedom in their bids in the form of an intercept.

In summary, our contributions to this paper include the following.

- We present a mathematical model for each of the three market mechanisms. We prove the existence of Nash equilibria and derive analytical expressions of the equilibria for symmetric firms.
- We show that the stochastic settlement mechanism with fixed deviation penalties can outperform the NZTS model given that the deviation penalty is chosen appropriately, i.e., in a (large) interval that includes the true deviation costs.
- To investigate the asymmetric case, through an iterative process, we compute the equilibria of the models numerically. Again, the stochastic settlement outperforms the NZTS model.
- We investigate a special case of stochastic settlement with zero intercepts. We find the unique equilibrium and show that the previous conclusion cannot be extended to this restricted version of ISOSP.

2. The Market Environment

In this paper, we aim to compare different market designs for electricity. We begin by presenting assumptions that are common to all markets we consider, features of what we call the *market environment*. These include such considerations as the number of firms, the costs firms face, the structure of demand, and so forth.

Assumption 1. *The market environment may be defined by the following features.*

- *Electricity is traded over a network with no transmission constraints and no line losses, thus we may consider all trading as taking place at a single node.*
- *Demand for electricity is uncertain, and may realize in one of $s \in \{1, \ldots, S\}$ possible outcomes (scenarios), each with probability θ_s. Demand in state s is assumed to be linear, and defined by the inverse demand function $p_s = Y_s - ZC_s$, where C_s is the quantity of electricity and p_s is the market price, in scenario s, and Z is the slope that indicates the rate of change of price as a function of quantity C_s; all scenarios have the same slope. Without loss of generality, assume $Y_1 < Y_2 < \cdots < Y_S$. We will denote the expected value of Y_s by $Y = \sum_s \theta_s Y_s$. The distribution of demand is common knowledge to all agents.*
- *There are n identical firms wishing to sell electricity.*
- *For a given firm i in scenario s, we will denote the predispatch quantity by q_i, and any short-run change in production by $x_{i,s}$. Variable q_i is a "here and now" decision*

made by the ISO and $x_{i,s}$ is a "wait and see" decision to adjust production to meet demand. Thus a generator's actual production in scenario s is equal to $q_i + x_{i,s}$, which we denote by $y_{i,s}$.

- *To produce $y_{i,s}$, firm i will incur a cost of production given by $\alpha(q_i + x_{i,s}) + (\beta/2)(q_i + x_{i,s})^2$ plus a deviation cost (from the targeted q_i production), $(\delta/2)x_{i,s}^2$. Here q_i is the expected dispatch of firm i, and $q_i + x_{i,s}$ is the actual short-run dispatch and $\delta > 0$.*
- *As minimum marginal cost of generation should not be more than maximum price of electricity, we assume*

$$\alpha \leqslant Y_s \quad \forall s \in \{1, \ldots, S\}.$$

- *There is an ISO who takes bids and determines dispatch and prices according to the given market design.*
- *All agents have full information about costs and the distribution of demand (scenarios and probabilities).*
- *We assume that the strategy space of each participant is defined by their choice of linear supply function parameters discussed under each market clearing mechanism.*

Our assumptions on generators' cost functions are particularly critical to the analysis that follows, and deserve further explanation. Generators face two distinct costs when generating electricity. If given sufficient advance notice of the quantity they are to dispatch, the generator can plan the allocation of turbines to produce that quantity most efficiently. This is what we mean by a *long-run* cost function. The interpretation of this is the lowest possible cost at which a generator can produce quantity q. In electricity markets, however, demand fluctuates at short notice, and the ISO may ask a generator to change its dispatch at short notice. In this case, generators may not have enough time to efficiently reallocate their turbines. For example, many thermal turbines take hours to ramp-up. Most likely, the generator will have to adopt a less efficient production method, such as running some turbines above their rated capacity, which also increases the wear on the turbines. Thus there is some inherent cost in deviating from an expected predispatch in the short-run. This cost can be incurred even if the requested deviation is negative. We assume that the generator will be unable to revert to the most efficient mode of producing this quantity $q_i + x_{i,s}$ in the short-run, so it pays a penalty cost. Note that this imposes a positive penalty cost upon the generator for making the short-run change, even if the change is negative. This penalty cost is additively imposed on top of the "efficient" cost of producing at the new level. We call this cost the *deviation cost*. Note that we assume the symmetric case in which cost of generation and deviation is determined through the same constant parameters (α, β, δ).

Our goal is to compare the outcomes of different markets imposed upon this environment. To be able to draw comparisons in different paradigms, we need to examine the steady state behaviour of participants under the different market clearing mechanisms. To this end, we need to compute equilibria arising under the different market clearing mechanisms. To make the computations tractable, we will restrict the firms to offer linear supply functions in the following sections of this paper.

3. Two-Period Settlement (NZTS) Model

In this section, we will introduce a two-period market that is inspired by the market clearing mechanism as it operates currently in New Zealand. As explained in the introduction, in the NZEM firms bid a step supply function for a given half hour period. The bid is made at least two hours in advance. Once gate closure occurs (two hours in advance of any given period),the supply function offers can not be changed. (This is at least in part because of the fact that there is no capacity market in the NZEM and the system operator may be faced with a real possibility of curtailing demand if the generators were to change their bids in real time.) The market will then be cleared six times, every five minutes during the given half-hour period. Each five-minute redispatch is computed with real-time demand, but with the supply offer stacks that have been submitted prior to gate closure.[3] We simplify the situation by assuming the market clears only twice; once after the offers are submitted, but before demand is realized. We call this the "predispatch" phase, which tells the generators approximately how much they should produce. Once demand is realized, the same offers will be used to determine actual dispatch in what we call the "spot settlement." The difference between predispatch and spot dispatch is a generator's short-run deviation, which is subject to potentially higher costs as we described earlier. However, the ISO has no knowledge of this cost and it is not explicitly stated in the generators' bids. This cost can be indirectly reflected in the supply functions the generators bid in.

3.1. Mathematical Model

Our simplified model for the NZTS market has two distinct stages: predispatch and spot. Following the large body of literature on affine supply functions (see, e.g., Green 1996, Baldick et al. 2004) we will work with a linear demand curve and frame generator supply offers as linear functions. Explicitly, each generator i bids a supply function $a_i + b_i q_i$ before the predispatch market to represent their quadratic costs. This supply function is required to be increasing, i.e., the offered b_i must satisfy $b_i \geqslant \varepsilon > 0$, where ε is the machine accuracy.[4] Note that unlike (Green 1996), we do not assume that the intercept $a_i = 0$. Following our main analysis, we briefly present a special case where $a_i = 0$ is required.

When generators lock in their offers, demand is uncertain (in New Zealand this point in time is referred to as *gate closure*). The ISO will then use the generator's bid twice: once to clear the predispatch market, and once again after demand is realized to clear the spot market. As in reality, in both the predispatch and spot markets, the ISO aims to maximize social welfare, assuming generators are bidding their true cost functions. Since demand is unknown in predispatch, the ISO will nominate (and use) an expected demand (and will not consider the distribution of demand):

$$\min_{q_i,\, i\in\{1,\ldots,n\}} z, \quad z = \sum_{i=1}^{n}\left[\left(a_i q_i + \frac{b_i}{2}q_i^2\right) - \left(YQ - \frac{Z}{2}Q^2\right)\right]$$
$$\text{s.t. } \sum_{i=1}^{n} q_i - Q = 0 \quad [f]. \tag{1}$$

From this first settlement, the ISO can extract a forward price f equal to the shadow price on the (expected demand balance) constraint. f is not used for any settlements as the predispatch quantity and prices are merely a guide at this stage. Recall that the predispatch quantity for generator i is denoted by q_i. After predispatch is determined, true demand is realized, and the ISO then clears the spot market (using the specific demand scenario that has been realized) to maximize welfare by solving (2):

$$\min_{y_{i,s}} z, \quad z = \sum_{i=1}^{n}\left[\left(a_i y_{i,s} + \frac{b_i}{2}y_{i,s}^2\right) - \left(Y_s C_s - \frac{Z}{2}C_s^2\right)\right]$$
$$\text{s.t. } \sum_{i=1}^{n} y_{i,s} - C_s = 0 \quad [p_s]. \tag{2}$$

Here again the ISO computes a spot price p_s as the shadow price on the constraint. (Note that we can eliminate the constraint and substitute C_s in the objective, however, imposing this constraint enables the easy introduction of the price as the shadow price attached to the constraint.) The generator is not permitted to change its bid after predispatch, but does face an additional deviation cost δ for its short-run deviation.

Note that in both ISO optimization problems (1)–(2) we have dispensed with nonnegativity constraints on the predispatch and dispatch quantity both, following the convention of supply function equilibrium models (see, e.g., Klemperer and Meyer 1989, Bolle 1992), in order to enable the analytic computation of equilibria. We will demonstrate that the resulting symmetric equilibria of our NZTS market model will always have associated nonnegative predispatch and dispatch quantities.

Firm i's profit in scenario s in this market is then given by

$$u_{i,s}(q_i, y_{i,s}) = p_s y_{i,s} - \left(\alpha y_{i,s} + \frac{\beta}{2}y_{i,s}^2 + \frac{\delta}{2}(y_{i,s} - q_i)^2\right). \tag{3}$$

3.2. Equilibrium Analysis of the Two-Period Market

In this section, we will present equilibria of the NZTS market model. We will first compute the optimal dispatch quantities from the ISO's optimal dispatch problems (1) and (2) for any number of players. We will then embed these quantities in each generator's expected profit function and allow the generators to simultaneously optimize over their (linear) supply function parameters to obtain equilibrium offers.

Proposition 1. *Problem (1) is a convex program with a strictly convex objective. Its unique optimal solution and the corresponding optimal dual f are given by*

$$f = \frac{Y + ZA}{ZB + 1}, \quad q_i = fB_i - A_i,$$

where $A_i = a_i/b_i$, $B_i = 1/b_i$, $A = \sum_{i=1}^{n} A_i$ and $B = \sum_{i=1}^{n} B_i$.

Proof. Note that problem (1) has a single linear constraint and that its objective is a strictly convex quadratic as we have assumed that $b_i > 0$ and $Z > 0$. The problem therefore has a unique optimal solution delivered by the first-order conditions provided below:

$$Q - \sum_{i=1}^{n} q_i = 0, \tag{4}$$
$$f - Y + ZQ = 0, \tag{5}$$
$$-f + a_i + b_i q_i = 0 \quad i \in \{1, 2, \ldots, n\}. \tag{6}$$

Algebraic manipulation of Equations (5)–(6) will provide the results. □

We note that Proposition 1 (and the next proposition) are similar to analysis in Green (1996, 1999), although we allow for an intercept parameter in our model.

Proposition 2. *For each scenario s, problem (2) is a convex program with a strictly convex objective. Its unique optimal solution and the corresponding optimal dual p_s are given by*

$$p_s = \frac{Y_s + ZA}{ZB + 1}, \quad y_{i,s} = p_s B_i - A_i,$$

where A_i, B_i, A, and B are defined in Proposition 1.

Proof. Problems (2) and (1) are structurally identical, therefore the simple proof of Proposition (1) applies again here. □

Remark 1. Note from the above that the predispatch price (and quantity) are equal to the expected spot market prices (and quantities, respectively). That is,

$$f = \sum_{s=1}^{S} \theta_s p_s. \tag{7}$$

We will now compute the linear supply functions resulting from the equilibrium of the TS market game laid out in (1).

3.2.1. Firm i's Computations. In this section we will focus on firm i's expected profit function. Note that using Equation (7) we obtain

$$\begin{aligned} u_i &= E_s[u_{i,s}] \\ &= \sum_{s=1}^{S} \theta_s \left(p_s y_{i,s} - \left(\alpha y_{i,s} + \frac{\beta}{2} y_{i,s}^2 + \frac{\delta}{2}(y_{i,s} - q_i)^2 \right) \right). \end{aligned}$$

Using Propositions 1 and 2, we can rewrite u_i as a function of a_i and b_i. To find a maximum of u_i (for a fixed set of competitor offers) we appeal to a transformation that will yield concavity results for u_i. We consider u_i to be a function of A_i and B_i (instead of a_i and b_i). Note that the transformation ($A_i = a_i/b_i, B_i = 1/b_i$) is a one-to-one transformation.

Proposition 3. *Let all competitor (linear) supply function offers be fixed. The following maximizes u_i (and is therefore firm i's best response):*

$$B_i = \frac{1 + ZB_{-i}}{Z + \beta + \delta + Z(\beta + \delta)B_{-i}}$$
$$A_i = \frac{\alpha + B_i(Z\alpha - \delta(Y + ZA_{-i})) + Z\alpha B_{-i}}{2Z + \beta + Z\beta B_{-i}},$$

where $A_{-i} = \sum_{j \neq i} A_j$ and $B_{-i} = \sum_{j \neq i} B_j$.

Proof. We can show that u_i is a concave function of A_i, assuming B_i is a fixed parameter. Here we have dispensed with the expression for u_i as a function of A_i and B_i as it is long and rather complicated. We note that u_i is a smooth function of A_i and B_i and that

$$\frac{\partial^2 u_i}{\partial A_i^2} = -\frac{(1 + ZB_{-i})(2Z + \beta + Z\beta B_{-i})}{(1 + ZB)^2} \leqslant 0.$$

Let B_i be arbitrary but fixed. As u_i is a concave function of A_i the first-order condition yields an expression for $A_i^*(B_i)$, the value of A_i that maximizes u_i (for the fixed B_i):

$$\begin{aligned} A_i^*(B_i) = \big[&(1 + ZB_{-i})(-Y + \alpha - ZA_{-i} + Z\alpha B_{-i}) \\ &+ B_i(Z(Y + ZA_{-i}) + (Z\alpha + \beta Y + Z\beta A_{-i}) \\ &\cdot (ZB_{-i} + 1))\big] \cdot \big[(1 + ZB_{-i})(2Z + \beta + Z\beta B_{-i})\big]^{-1}. \end{aligned}$$

We can embed $A_i^*(B_i)$ into u_i and find the maximizer in terms of B_i. This is enough to demonstrate that the end result delivers the maximum of u_i.

After embedding this value of A_i^* into the profit function, the derivative with respect to B_i of u_i is

$$\frac{\partial u_i}{\partial B_i} = \frac{(Y^2 - \sum_s \theta_s Y_s^2)(-1 + (Z + \beta + \delta)B_i + Z(-1 + (\beta + \delta)B_i)B_{-i})}{(1 + ZB)^3}.$$

The zero of this derivative is $B_i^* = (1 + ZB_{-i})/(Z + \beta + \delta + Z(\beta + \delta)B_{-i})$. Recall that $Y = \sum_s \theta_s Y_s$, therefore Jensen's inequality implies $Y^2 - \sum_s \theta_s Y_s^2 \leqslant 0$. Thus, $\partial u_i/\partial B_i \geqslant 0$, when $B_i < B_i^*$, and $\partial u_i/\partial B_i \leqslant 0$, when $B_i > B_i^*$. In other words, u_i is a quasiconcave function of B_i and is maximized at $B_i = B_i^*$.

Note that evaluating A_i^* at B_i^* yields

$$A_i^* = \frac{\alpha + B_i(Z\alpha - \delta(Y + ZA_{-i})) + Z\alpha B_{-i}}{2Z + \beta + Z\beta B_{-i}}. \quad \square$$

From the above, we can obtain the equilibrium of the NZTS model by solving all best responses simultaneously. This gives the unique and symmetric solution 2S-EQM:

$$\begin{aligned} B_i = 2\big[&-(n-2)Z + \beta + \delta \\ &+ \sqrt{(n-2)^2 Z^2 + 2nZ(\beta + \delta) + (\beta + \delta)^2}\,\big]^{-1} \end{aligned} \tag{8}$$

$$A_i = \frac{\alpha + (nZ\alpha - Y\delta)B_i}{2Z + \beta + (n-1)Z(\beta + \delta)B_i}, \tag{9}$$

or alternatively
2S-EQM:

$$\begin{aligned} b_i = \tfrac{1}{2}\big[&-(n-2)Z + \beta + \delta \\ &+ \sqrt{(n-2)^2 Z^2 + 2nZ(\beta + \delta) + (\beta + \delta)^2}\,\big] \end{aligned} \tag{10}$$

$$a_i = \frac{\alpha b_i + (nZ\alpha - Y\delta)}{2Zb_i + \beta b_i + (n-1)Z(\beta + \delta)}. \tag{11}$$

As we discussed earlier, these equilibrium offers yield nonnegative predispatch and dispatch quantities. Below we formally state this result.

Proposition 4. *The equilibrium predispatch and spot production quantities of the firms in the NZTS market are nonnegative, i.e., $q_i \geqslant 0 \ \forall\, i$, and $y_{i,s} \geqslant 0 \ \forall\, i, s$ where q_i and $y_{i,s}$ are the optimal solutions to problems (1) and (2), respectively, using the equilibrium parameters from (10) and (11).*

Proof. For the proof, please consult the e-companion. $\square$

4. Stochastic Settlement Market

4.1. ISOSP Model

We now introduce the market model we will use to analyze a stochastic settlement market. As discussed in the introduction, the stochastic settlement market contains only a single stage of bidding, but the market clearing procedure takes into account the distribution of future demand when determining dispatch. The market works as follows. When the market opens, demand is uncertain. Firms are allowed to bid their "normal" cost functions (the cost of producing a given output most efficiently) and a "penalty" cost function that they would need to be paid to deviate in the short-run. Since firms have quadratic cost functions, they can bid their true costs by submitting a linear supply function. Each firm i chooses a_i and b_i to bid the linear supply function $a_i + b_i q$, and d_i to bid the marginal penalty cost applied to the deviation from predispatch quantity. For a marginal deviation penalty, we look for a function that is zero when the dispatch is not changed from the predispatch quantity, as well as positive to the right and negative to the left of the predispatch. One of the simplest forms this function can take is the linear form we have assumed. While the true marginal cost of deviation for a station may be nonlinear, it is expected to be smooth as it relates to engineering attributes such as flow of water through an aperture. Therefore to the first order, it can be approximated by a linear function. Note that as with the NZTS model, these bids (a_i, b_i, d_i) need not be their true values (α, β, δ). The offered b_i is required to be positive and d_i should be nonnegative.

After generators have placed their bids, the ISO computes the market dispatch according to the stochastic settlement model (outlined below). At this point demand is still uncertain. The ISO chooses two key variables. The first is the *predispatch* quantity for each firm. This is the quantity the ISO asks each firm to prepare to produce, namely, the predispatch quantities q_i defined in Section 2. The second is the *short-run deviation* for generator i under each scenario s. This deviation is the variable $x_{i,s}$ defined in Section 2, representing the adjustment made to firm i's predispatch quantity in scenario s. The ISO can choose both predispatch and short-run deviations simultaneously, while aiming to maximize expected social welfare. The ISO assumes that generators have bid their true costs.

In the final stage, demand is realized, and the ISO will ask generators to modify their predispatch quantity according to the short-run deviation for the particular scenario. Each generator ends up producing $q_i + x_{i,s}$. Two prices are calculated during the course of optimizing welfare. The first is the (shadow) price of the predispatch quantities. We will denote this by f. The second are the prices of each of the deviations, for each of the scenarios. We will denote these by p_s for scenario s. Each generator is paid f per unit for its predispatch quantity q_i, and p_s for its deviations $x_{i,s}$. Thus in realization s, generator i makes profit equal to

$$u_{i,s}(q_i, y_{i,s}) = p_s y_{i,s} - \left(\alpha y_{i,s} + \frac{\beta}{2} y_{i,s}^2 + \frac{\delta}{2}(y_{i,s} - q_i)^2\right). \quad (12)$$

The stochastic optimization problem solved by the ISO can be represented as follows.[5]

ISOSP:

$$\begin{aligned}
\min_{q_i, x_{i,s}} \ & \sum_{s=1}^{S} \theta_s \left(\sum_{i=1}^{n} \left[a_i(q_i + x_{i,s}) + \left(\frac{b_i}{2}\right)(q_i + x_{i,s})^2 \right.\right. \\
& \left.\left. + \left(\frac{d_i}{2}\right) x_{i,s}^2 \right] - \left(Y_s C_s - \left(\frac{Z}{2}\right) C_s^2 \right) \right) \\
\text{s.t.} \ & \sum_i q_i - Q = 0 \quad [f] \\
& Q + \sum_i x_{i,s} - C_s = 0 \quad \forall s \in \{1, \dots, S\} \quad [\theta_s p_s],
\end{aligned}$$

where Q and C_s stand for the total predispatched quantity and total consumption in scenario s, respectively. Note that we could have eliminated the two equality constraints. However, their dual variables inform the market prices f and p_s, respectively, so for clarity we have left them in. Here p_s is the probability adjusted shadow price for the second stage.

Note that in the ISOSP, an estimate of a future distribution is used. Khazaei et al. (2013) examine the effectiveness of ISOSP in an empirical competitive setting.

4.2. Characteristics of the Stochastic Optimization Problem

We begin by presenting a series of results that simplify the set of solutions to the ISOSP problem. We start by establishing technical lemmas that enable us to prove that our ISOSP is equivalent to a two-period market clearing mechanism similar to NZTS, with the essential difference that now a deviation penalty is present in the ISO's dispatch in real time. These results drastically simplify the subsequent analysis of firms' behaviour in equilibrium.

Lemma 1. *In the stochastic settlement market clearing, the expected deviation of firm i from predispatch quantity q_i^* is zero. That is, the optimal solution to ISOSP will always satisfy*

$$\sum_s \theta_s x_{i,s}^* = 0.$$

Proof. Let us assume q_i^* and $x_{i,s}^*$ form ISOSP's optimal solution. Let us define for each i and s the quantity $k_{i,s} = q_i^* + x_{i,s}^*$, the total production of firm i in scenario s. Note that $C_s = \sum_i q_i^* + \sum_i x_{i,s}^*$. Assume, on the contrary, that there exists at least one firm j such that $\sum_s \theta_s x_{j,s}^* \neq 0$. The optimal objective value of ISOSP is then given by

$$\begin{aligned}
& \sum_i \sum_s \theta_s \left(a_i k_{i,s} + \frac{b_i}{2}(k_{i,s})^2 \right) + \sum_i \sum_s \theta_s \frac{d_i}{2}(x_{i,s}^*)^2 \\
& + Y_s \sum_i k_{i,s} - \frac{Z}{2}\left(\sum_i k_{i,s} \right)^2. \quad (13)
\end{aligned}$$

Note that as $\sum_s \theta_s x_{j,s}^* \neq 0$, the term $\sum_i \sum_s \theta_s (d_i/2)(x_{i,s}^*)^2$ is positive. Now, for a fixed i and $k_{i,s}$ given from above, consider the problem

$$\begin{aligned}
& \min_{q_i, x_{i,s}} \frac{d_i}{2} \sum_{s=1}^{S} \theta_s x_{i,s}^2 \\
& q_i + x_{i,s} = k_{i,s} \quad \forall s \in \{1, \dots, S\}.
\end{aligned} \quad (14)$$

This problem clearly reduces to the univariate problem

$$\min_{q_i} \sum_{s=1}^{S} \theta_s (k_{i,s} - q_i)^2,$$

which is optimized at

$$q_i = \sum_{s=1}^{S} \theta_s k_{i,s}.$$

We will now construct a new optimal solution for ISOSP, with an improved objective value and hence derive a contradiction. Define $\hat{q}_i$ and $\hat{x}_{i,s}$ by

$$\hat{q}_i = \begin{cases} q_i^*, & i \neq j \\ \sum_{s=1}^{S} \theta_s k_{j,s} & \text{otherwise,} \end{cases}$$

and

$$\hat{x}_{i,s} = \begin{cases} x_{i,s}^*, & i \neq j \\ k_{j,s} - \hat{q}_j & \text{otherwise.} \end{cases}$$

By definition, $\hat{q}_i + \hat{x}_{i,s} = q_i^* + x_{i,s}^*$ for all i and s. It is easy to see that the quantities $\hat{q}_i$ and $\hat{x}_{i,s}$ yield a feasible solution to ISOSP by satisfying $\sum_{s=1}^{S} \hat{q}_i + \hat{x}_{i,s} = C_s$. Furthermore note that $\hat{x}_{j,s} = x_{j,s}^* - \sum_s \theta_s x_{j,s}^*$. Hence $\sum_s \theta_s (\hat{x}_{j,s})^2 < \sum_s \theta_s (x_{j,s}^*)^2$ since the expected value of $\hat{x}_{j,s}$ is zero.

Now, the objective function evaluated at $\hat{q}_i$ and $\hat{x}_{i,s}$ is given by

$$\sum_i \sum_s \theta_s \left(a_i k_{i,s} + \sum_i \sum_s \theta_s \frac{d_i}{2} (\hat{x}_{i,s})^2 + \frac{b_i}{2} (k_{i,s})^2 \right) + Y_s \sum_i k_{i,s} - \frac{Z}{2} \left(\sum_i k_{i,s} \right)^2.$$

This value is strictly less than the objective evaluated at q_i^* and $x_{i,s}^*$ (given by (13)), as we have already established that $\sum_i \sum_s \theta_s (d_i/2)(x_{i,s}^*)^2 > \sum_i \sum_s \theta_s (d_i/2)(\hat{x}_{i,s})^2$. This yields the contradiction that proves the result. □

Corollary 1. *In the stochastic problem ISOSP, if $q_i^* + x_{i,s}^* \geqslant 0$ is satisfied $\forall\, s \in \{1, \ldots, S\}$ then $q_i^* \geqslant 0$ will hold.*

Proof. In Lemma 1 we established that $\sum_s \theta_s x_{i,s}^* = 0$. Therefore there exists a scenario s' such that $x_{i,s'}^* \leqslant 0$. Clearly then $q_i^* + x_{i,s'}^* \geqslant 0$ implies $q_i^* \geqslant 0$. □

Discussion. Lemma 1 is the crucial result that drives the rest of our characterizations. This result hinges on the fact that we penalize quadratic deviation from the predispatch quantity. In the proof of Lemma 2, we demonstrate that the second stage of ISOSP reduces to selecting a contract point that minimizes the quadratic deviation penalty function that is known to be the mean of any distribution. For the quadratic penalty, this is irrespective of the distribution. (It is possible to also use an absolute value based deviation penalty and require a symmetric demand distribution. When the penalty function is an absolute value, the point of best estimate is the median of the distribution. For a symmetric distribution, of course this reduces again to the mean.) This model penalizes the deviations upward and downward identically. Therefore the predispatch point is optimized based on the mean demand scenario. The reader may argue that allowing for different upward and downward penalties is more realistic. However, as Pritchard et al. (2010) show, such allowance of asymmetric penalties can lead to systematic arbitrage by the ISO, where a generator may be required to deviate upward "in every scenario" simply to increase expected welfare. This may be deemed undesirable for a market clearing mechanism. For this reason and to aid analytical computations, we have confined our attention to the symmetric upward and downward penalty case for this paper, which guarantees systematic arbitrage will not occur. We now use the above results to prove that the ISO's optimization problem can be viewed as a two-period settlement system where unlike NZTS, the deviation penalties are explicitly stated in the ISO's problem in the second period.

Lemma 2. *The objective function of ISOSP is equivalent to the following function, which is separable in the predispatch and the spot market variables*

$$z = \sum_{i=1}^{n} \left(a_i q_i + \frac{b_i}{2} q_i^2 \right) - Y \sum_{i=1}^{n} q_i + \frac{Z}{2} \left(\sum_{i=1}^{n} q_i \right)^2 + \sum_{i=1}^{n} \left(\frac{b_i + d_i}{2} \sum_{s=1}^{S} \theta_s x_{i,s}^2 \right) - \sum_{i=1}^{n} \sum_{s=1}^{S} \theta_s Y_s x_{i,s} + \frac{Z}{2} \sum_{s=1}^{S} \theta_s \left(\sum_{i=1}^{n} x_{i,s} \right)^2.$$

Proof. Substituting for C_s from constraints into the objective function of ISOSP yield

$$z = \sum_{i=1}^{n} \left(a_i q_i + \frac{b_i}{2} q_i^2 \right) - Y \sum_{i=1}^{n} q_i + \frac{Z}{2} \left(\sum_{i=1}^{n} q_i \right)^2 + \sum_{i=1}^{n} \left(\frac{b_i + d_i}{2} \sum_{s=1}^{S} \theta_s x_{i,s}^2 \right) - \sum_{i=1}^{n} \sum_{s=1}^{S} \theta_s Y_s x_{i,s} + \frac{Z}{2} \sum_{s=1}^{S} \theta_s \left(\sum_{i=1}^{n} x_{i,s} \right)^2 + \sum_{i=1}^{n} \left(a_i \sum_{s=1}^{S} \theta_s x_{i,s} \right) + \sum_{i=1}^{n} \left(q_i b_i \sum_{s=1}^{S} \theta_s x_{i,s} \right) + \sum_{s=1}^{S} \theta_s Z \sum_{i=1}^{n} \sum_{j=1}^{n} q_i x_{j,s}.$$

We have split the objective in three parts above. Note that the first part of the objective above is exclusively a function of predispatch quantities q_i and the second only a function of the spot dispatches $x_{i,s}$. The third segment can be eliminated at optimality since in Lemma 1 we proved that $\sum_{s=1}^{S} \theta_s x_{j,s} = 0$ for the optimal choice of real-time dispatches. □

Note. We have therefore established that ISOSP reduces to a two-period single settlement model very similar to NZTS but with penalties d_i explicitly present in the second period.

The rest of this section is devoted to deriving explicit expressions for the solution of ISOSP. In the next section we will use these expressions to arrive at best response functions for the firms and subsequently in constructing an equilibrium for the stochastic market settlement. To simplify the equations and arrive at explicit solutions, we will transform the space of the parameters of ISOSP (i.e., the firm decision variables), much in the same way that we did in Section 3. If we now define $R_i = 1/(b_i + d_i)$ and $R = \sum_i R_i$, ISOSP reduces to minimizing the following:

$$\sum_{i=1}^{n}\left(\frac{A_i}{B_i}q_i + \frac{1}{2B_i}q_i^2\right) - Y\sum_{i=1}^{n} q_i + \frac{Z}{2}\left(\sum_{i=1}^{n} q_i\right)^2 + \sum_{s=1}^{S}\theta_s\left[\sum_{i=1}^{n}\frac{1}{2R_i}x_{i,s}^2 - (Y_s - Y)\sum_{i=1}^{n} x_{i,s} + \frac{Z}{2}\left(\sum_{i=1}^{n} x_{i,s}\right)^2\right].$$

Note that as before (Lemma 2), the problem is separable in q_i's and $x_{i,s}$'s, we can therefore solve the two stages separately. Note also that the problem in each stage is a convex optimization problem, therefore the first-order conditions will readily produce the optimal solution.

Proposition 5. *If (q, x, f, p) represents the solution of ISOSP, then we have*

$$q_i = \frac{(Y + ZA)B_i}{1 + ZB} - A_i, \tag{15}$$

$$x_{i,s} = \frac{(Y_s - Y)R_i}{1 + ZR}, \tag{16}$$

$$f = \frac{Y + ZA}{1 + ZB}, \quad p_s = \frac{Y + ZA}{1 + ZB} + \frac{Y_s - Y}{1 + ZR}.$$

Proof. For derivation of the expressions for the optimal solution above from first-order conditions please refer to the e-companion. □

Observe from the expression for f that this forward price (paid on predispatch quantities) is independent of any deviation costs in the spot market.

Corollary 2. *In the solution of ISOSP, forward price is equal to the expected spot market price.*

Proof. Since $Y = \sum_s \theta_s Y_s$, it is immediately obvious that $f = \sum_s \theta_s p_s$ as required. □

4.3. Equilibrium Analysis of the Stochastic Settlement Market (ISOSP)

In Section 4.1 we presented firm i's profit under scenario s in Equation (12). In our market model, we assume that all firms are risk neutral and therefore interested only in maximizing their expected profit. Firm i's expected profit (using the results of Corollary 2) is given by

$$u_i = fq_i + \sum_{s=1}^{S}\theta_s\left(p_s x_{i,s} - \left(\alpha(q_i + x_{i,s}) + \frac{\beta}{2}(q_i + x_{i,s})^2 + \frac{\delta}{2}x_{i,s}^2\right)\right). \tag{17}$$

The above expression for u_i can be expanded and we can observe that

$$u_i = fq_i - \left(\alpha q_i + \frac{\beta}{2}q_i^2\right) + \sum_{s=1}^{S}\theta_s\left(p_s x_{i,s} - \frac{\beta+\delta}{2}x_{i,s}^2\right) - \alpha\sum_{s=1}^{S}\theta_s x_{i,s} - \beta q_i\sum_{s=1}^{S}\theta_s x_{i,s}.$$

Note that from Lemma 1, the generator would know that for any admissible bid, the corresponding expected deviation from predispatch quantities $\sum_{s=1}^{S}\theta_s x_{i,s} = 0$. Therefore the expected profit for the generator becomes

$$u_i = fq_i - \left(\alpha q_i + \frac{\beta}{2}q_i^2\right) + \sum_{s=1}^{S}\theta_s\left(p_s x_{i,s} - \frac{\beta+\delta}{2}x_{i,s}^2\right).$$

We can use the expressions obtained from Proposition 5 to write u_i as follows:

$$\begin{aligned} u_i = {} & -\frac{1}{2}\beta A_i^2 + \frac{A_i(-ZA + \alpha + ZB\alpha + ZA\beta B_i + Y(-1 + \beta B_i))}{1 + ZB} \\ & + \frac{1}{2(1+ZB)^2(1+ZR)^2}\Big(2(1+ZR)^2(ZA+Y) \\ & \cdot (ZA + Y - (1+ZB)\alpha)B_i - (1+ZR)^2(ZA+Y)^2\beta B_i^2 \\ & + (1+ZB)^2 R_i(-2 + (\beta+\delta)R_i)\Big(Y^2 - \sum_s \theta_s Y_s^2\Big)\Big). \end{aligned} \tag{18}$$

Although this expression of the expected profit for the generator is rather ugly, it does have the advantage that upon differentiating with respect to R_i, all dependence on A_i and B_i drops and we are left with

$$\frac{\partial u_i}{\partial R_i} = \left(\left(Y^2 - \sum_s \theta_s Y_s^2\right)(-1 + (Z+\beta+\delta)R_i + ZR_{-i}(-1 + (\beta+\delta)R_i))\right)\cdot\frac{1}{1+ZR^3}, \tag{19}$$

where $R_{-i} = \sum_{j\neq i} R_j$. The fact that this derivative is free of A_i and B_i indicates that u_i is separable in R_i and (A_i, B_i), that is

$$u_i(A_i, B_i, R_i) = g_i(A_i, B_i) + h_i(R_i). \tag{20}$$

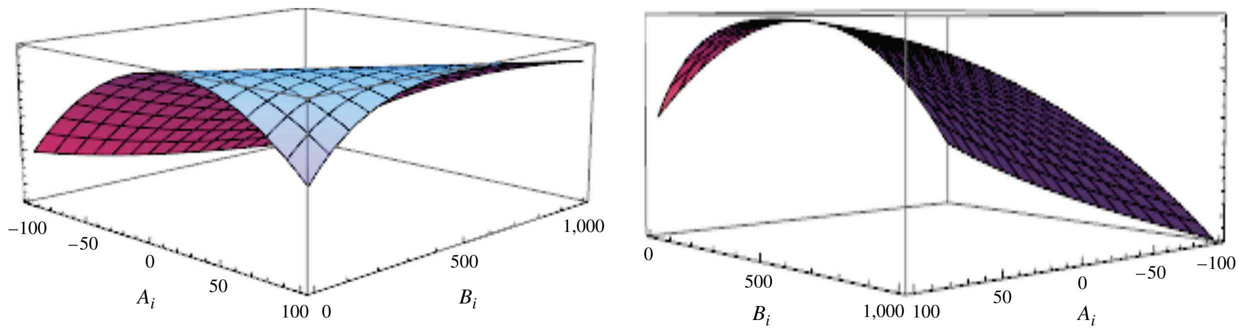
Because of this natural separability, our equilibrium analysis will focus on finding best responses in terms of A_i, R_i, and B_i, very similar to the NZTS section.

Equation (20) enables us to maximize u_i by maximizing g_i and h_i over (A_i, B_i) and R_i, respectively. This is helpful as we can establish quasiconcavity results for g_i and h_i separately.

We start our investigations by examining g_i. The full expression for g_i can be found in the e-companion. Holding B_i fixed, note that

$$\frac{\partial^2 g_i}{\partial A_i^2} = -\frac{(1+ZB_{-i})(2Z+\beta+Z\beta B_{-i})}{(1+ZB)^2}.$$

Figure 1. (Color online) Two Views of the Graph of Function g_i



Note. Note that the optimal value of g_i is obtained along a continuum, for any value of B_i.

This demonstrates that g_i is concave in A_i for any fixed B_i. Furthermore, for any fixed B_i, we can use the first-order conditions to find $A_i^*(B_i)$, i.e., the value of A_i that maximizes $g_i(A_i, B_i)$ for the fixed B_i:

$$A_i^*(B_i) = \big[(1+ZB_{-i})(\alpha - ZA_{-i} + Z\alpha B - Y) + (Y + ZA_{-i})(Z+\beta+Z\beta B_{-i})B_i\big] \cdot \big[(1+ZB_{-i})(2Z+\beta+Z\beta B_{-i})\big]^{-1}. \tag{21}$$

To find the optimal value for g_i, we substitute the expression for $A_i^*(B_i)$ in $g_i(A_i^*(B_i), B_i)$. Surprisingly, upon undertaking this substitution, it can be observed that $g_i(A_i^*(B_i), B_i)$ is a constant. Figure 1 depicts g_i.

To uncover the intuition behind this feature of g_i, we can offer the following mathematical explanation. We observe that

$$\begin{aligned}\frac{\partial g_i}{\partial A_i} = \big[&-(1+ZB_{-i})(Y - \alpha + ZA_{-i} + (2Z+\beta)A_i \\ &+ ZB_{-i}(-\alpha + \beta A_i))\big] \cdot \big[(1+ZB)^2\big]^{-1} \\ &+ \big[(Z(Y+\alpha) + Y\beta + Y(Z\alpha + Y\beta)B_{-i} \\ &+ ZA_{-i}(Z+\beta+Z\beta B_{-i}))B_i\big] \cdot \big[(1+ZB)^2\big]^{-1},\end{aligned}$$

and that

$$\frac{\partial g_i}{\partial B_i} = -\frac{Y + ZA}{1+ZB} \cdot \frac{\partial g_i}{\partial A_i}.$$

Therefore, stationary conditions enforced in A_i will also imply stationarity in B_i.

As $g_i(A_i^*(B_i), B_i)$ is constant for any $B_i > 0$, for any value of $B_i > 0$, the tuple $(A_i^*(B_i), B_i)$ is an argmax of g_i for any positive B_i. Let D_i denote $1/(d_i + \varepsilon)$. Recall that according to our initial assumptions, we have $b_i \geqslant \varepsilon$, thus $R_i \leqslant D_i$.

The following analysis on h_i will explain how optimal R_i is constrained by the value of D_i.

Proposition 6. *Suppose that R_{-i} is fixed. Then h_i is optimized at*

$$R_i^* = \min\left\{D_i, \frac{1+ZR_{-i}}{Z+\beta+\delta+Z(\beta+\delta)R_{-i}}\right\}.$$

Proof. Note that at

$$\hat{R}_i = \frac{1+ZR_{-i}}{Z+\beta+\delta+Z(\beta+\delta)R_{-i}}, \tag{22}$$

the derivative $\partial h_i/\partial R_i = \partial u_i/\partial R_i$ vanishes. Also recall from Jensen's inequality that $Y^2 \leqslant \sum_s \theta_s Y_s^2$. It can therefore be seen from (19) that this derivative is positive for $R_i < \hat{R}_i$ and negative for $R_i > \hat{R}_i$.

Recall further that the definitions of D_i and R_i require $R_i \leqslant D_i$. Therefore, in optimizing h_i, we need to enforce this constraint and we obtain

$$R_i^* = \min\left\{D_i, \frac{1+ZR_{-i}}{Z+\beta+\delta+Z(\beta+\delta)R_{-i}}\right\}. \quad \square$$

We now return to u_i, the expected profit function for firm i. As $u_i(A_i, B_i, R_i) = g_i(A_i, B_i) + h_i(R_i)$, we can start by obtaining the maximum value of g_i attained at a point $(A_i^*(B_i), B_i)$ for any positive B_i. Subsequently, we proceed to optimize $h_i(R_i)$. Proposition (6) readily delivers the optimal R_i. We have therefore proved the following theorem.

Theorem 1. *The best response of firm i, holding competitor offers fixed, is to offer any d_i for which we have*

$$\frac{1}{d_i+\varepsilon} \geqslant \frac{1+ZR_{-i}}{Z+\beta+\delta+Z(\beta+\delta)R_{-i}}.$$

For any such d_i, optimal a_i and b_i can be computed from the following equations:

$$\begin{aligned}R_i &= \frac{1+ZR_{-i}}{Z+\beta+\delta+Z(\beta+\delta)R_{-i}},\\ A_i &= \big[(1+ZB_{-i})(\alpha - ZA_{-i} + Z\alpha B - Y) + (Y+ZA_{-i})\\ &\quad \cdot (Z+\beta+Z\beta B_{-i})B_i\big] \cdot \big[(1+ZB_{-i})(2Z+\beta+Z\beta B_{-i})\big]^{-1}.\end{aligned}$$

Theorem 1 indicates that the game has multiple (infinite) symmetric equilibria.

4.4. Equilibrium Analysis of the Stochastic Settlement Market with Fixed Deviation Costs (ISOSP-FD)

Following Theorem 1, we are presented with an opportunity to assess a market clearing mechanism where the strategy space of the agents is reduced in dimension. Instead of bidding in parameters (a_i, b_i, d_i), in this variation the firms offer a supply function parametrised by (a_i, b_i) and the deviation penalty is fixed (and reflected in market clearing), by the ISO. We call this variation ISOSP-FD with $d_i = d^{SO}$ and known to the firms prior to the start of bidding. We show that for a range of d^{SO} starting anywhere above zero, building up to the true deviation cost and even beyond, the total welfare increases, in comparison to the NZTS model. This in turn demonstrates that having a deviation cost parameter, even though it may not be quite accurate, can enhance efficiency.

Proposition 7. *The unique symmetric equilibrium quantities of ISOSP-FD are as follows*:

$$
\begin{aligned}
b_i = \max\{\varepsilon, (-Z(n-2) + \beta + \delta + \sqrt{Z^2(n-2)^2} \\
+ 2Zn(\beta+\delta) + (\beta+\delta)^2) \cdot \tfrac{1}{2} - d^{SO}\} \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
a_i = \big[&\alpha - Y + B_i(-Z(Y(n-2) - (2n-1)\alpha) + Y\beta \\
&+ Z(n-1)(Zn\alpha + Y\beta)B_i)\big] \\
&\cdot \big[B_i(Z(n+1) + \beta + Y(n-1)(Zn+\beta)B_i)\big]^{-1}. \qquad (24)
\end{aligned}
$$

The proof of the above proposition is contained in the e-companion.

Let us define

$$
\hat{d} = \frac{-Z(n-2) + \beta + \delta + \sqrt{Z^2(n-2)^2 + 2Zn(\beta+\delta) + (\beta+\delta)^2}}{2}.
$$

Theorem 2. *For ISOSP-FD with $d^{SO} \leqslant \hat{d} - \varepsilon$, as the number of participating firms increases, they tend to offer their true cost parameters. In other words,*

$$
\lim_{n\to\infty} a_i = \alpha
$$
$$
\lim_{n\to\infty} b_i + d^{SO} = \beta + \delta.
$$

When the fixed parameter d^{SO} is chosen to be equal to δ, $\lim_{n\to\infty} b_i = \beta$.

Proof. The equations are simply derived from the equilibrium values of a_i and d_i given in Proposition 7. □

Theorem 2 shows that the ISOSP-FD market is behaving competitively in the sense that when number of firms increases, they offer their true cost parameters.

One important feature of the equilibrium values are the nonnegativity of the predispatch and dispatch. This is important, because we neglected the nonnegativity constraints in ISOSP in the first place.

Proposition 8. *Let $(\mathbf{q}^*, \mathbf{x}^*)$ represent an equilibrium of ISOSP-FD. Then the following inequalities hold*:

$$
\forall\, i, s: \quad q_i^* + x_{i,s}^* \geqslant 0
$$
$$
\forall\, i: \quad q_i^* \geqslant 0.
$$

The proof of the above proposition is contained in the e-companion.

Though, the equilibrium predispatch and dispatch are nonnegative, one might raise an objection that a game without the nonnegativity constraints embedded in the ISO's optimization problem, is different from the original game. Therefore, there is no assurance the found equilibrium is also the equilibrium of the original game. The following theorem states that the obtained equilibrium values are also the equilibrium of the original game with nonnegativity constraints. The proof of this theorem is quite lengthy and consists of several technical lemmas.

Theorem 3. *The equilibrium of the symmetric stochastic settlement game without the nonnegativity constraints in ISOSP-FD is also the equilibrium of the stochastic settlement game with the nonnegativity constraints.*

Proof. Please refer to the e-companion for the proof of this theorem. □

5. Comparison of the Two Markets

We are interested in the performance of the two market clearing mechanisms ISOSP-FD and NZTS, under strategic behaviour. Our criterion for comparing the two models is social welfare. Social welfare is defined as the sum of the consumer and producer welfare and in our market environments this reduces to

$$
\begin{aligned}
W = &\sum_{s=1}^{S} \theta_s \left(Y_s \left(\sum_{i=1}^{n} y_{i,s} \right) - \frac{Z}{2} \left(\sum_{i=1}^{n} y_{i,s} \right)^2 \right) \\
&- \sum_{s=1}^{S} \theta_s \left(\sum_{i=1}^{n} \left(\alpha y_{i,s} + \frac{\beta}{2} {y_{i,s}}^2 + \frac{\delta}{2} \left(y_{i,s} - q_i \right)^2 \right) \right). \qquad (25)
\end{aligned}
$$

Note that the different social welfare values W^{SS} (for the ISOSP-FD) and W^{NZTS} (for the NZTS mechanism) are found through the same formula, however with the different equilibrium $y_{i,s}$ variables.

Recall that following Theorem 1 the choice of d^{SO} was delegated to the ISO. The next theorem establishes that when firms are bidding strategically, the stochastic settlement market dominates the NZTS market for any choice of $d^{SO} \in (0, \hat{d})$.

Theorem 4. *The social welfare of ISOSP-FD is an increasing function of the parameter d^{SO}, and it is higher than the NZTS social welfare, provided d^{SO} is chosen less than or equal to the threshold value $\hat{d} - \varepsilon$.*

Proof. To prove the proposition, we show when $d^{SO}=0$, then $W^{SS}=W^{NZTS}$. We then demonstrate W^{SS} is an increasing function of d^{SO} for $0 \leqslant d^{SO} \leqslant \hat{d}-\varepsilon$, and therefore, $W^{SS} \geqslant W^{NZTS}$, when $d^{SO} \leqslant \hat{d}-\varepsilon$ (note that W^{NZTS} is a constant and does not change with d^{SO}).

When $d^{SO}=0$, Equations (24), (9), and (8) yield that the equilibrium quantities are identical in the stochastic settlement and two-period settlement markets. That is,

$$B_i^{SS}=B_i^{NZTS},\quad A_i^{SS}=A_i^{NZTS},\quad R_i^{SS}=B_i^{SS}.$$

Here we can simplify the expressions for $y_{i,s}$ and q_i (from Propositions 1, 2, and 5) to obtain

$$q_i^{SS}=q_i^{NZTS}=\frac{YB_i-A_i}{1+ZB},$$
$$y_{i,s}^{SS}=y_{i,s}^{NZTS}=\frac{Y_sB_i-A_i}{1+ZB}.$$

Therefore social welfare of these models (Equation (25)) are the same provided $d^{SO}=0$. We can rewrite the social welfare expression (25) as

$$W=\sum_{s=1}^{S}\theta_s\Bigg(Y_s\Bigg(\sum_{i=1}^{n}y_{i,s}\Bigg)-\frac{Z}{2}\Bigg(\sum_{i=1}^{n}y_{i,s}\Bigg)^2 -\sum_{i=1}^{n}\Bigg(\alpha y_{i,s}+\frac{\beta}{2}{y_{i,s}}^2+\frac{\delta}{2}{x_{i,s}}^2\Bigg)\Bigg). \tag{26}$$

Note that the expression for social welfare is the same for both models and only depends on the corresponding quantities dispatched from each model (i.e., $y_{i,s}^{SS}$ versus $y_{i,s}^{NZTS}$, etc.).

Note that, according to the results of Proposition 7, for $0\leqslant d^{SO}\leqslant \hat{d}-\varepsilon$, we have $b_i=\hat{d}-d^{SO}$, and therefore R_i has a constant value of $1/\hat{d}$, while B_i is a function of d^{SO}. Furthermore, note that $x_{i,s}^{SS}$ is independent of d^{SO}, and therefore

$$\frac{\partial W^{SS}}{\partial d^{SO}}=\frac{1}{(\hat{d}-d^{SO})^2}\sum_{i,s}\frac{\partial W^{SS}}{\partial y_{i,s}^{SS}}\frac{\partial y_{i,s}^{SS}}{\partial B_i}. \tag{27}$$

On the other hand, taking the derivative of $y_{i,s}^{SS}$ with respect to B_i we obtain

$$\frac{\partial y_{i,s}^{SS}}{\partial B_i}=\frac{(Y-\alpha)(n-1)Z^2}{(Z+nZ+\beta+(n-1)Z(nZ+\beta)B_i)^2}\geqslant 0.$$

The right-hand side is readily seen to be nonnegative as $Y>\alpha$ and $n>1$.

As $\partial y_{i,s}^{SS}/\partial B_i$ is independent of firm i and scenario s (note that $B_i=(\hat{d}-d^{SO})^{-1}$), we can rearrange (27) and obtain

$$\frac{\partial W^{SS}}{\partial d^{SO}}=\frac{1}{(\hat{d}-d^{SO})^2}\frac{\partial y_{i,s}^{SS}}{\partial B_i}\sum_{i,s}\frac{\partial W^{SS}}{\partial y_{i,s}^{SS}}.$$

On the other hand, differentiating (26) yields

$$\frac{\partial W^{SS}}{\partial y_{i,s}^{SS}}=\theta_s(Y_s-\alpha-(Zn+\beta)y_{i,s}^{SS}).$$

Hence

$$\sum_s\frac{dW^{SS}}{dy_{i,s}^{SS}}=Y-\alpha-(Zn+\beta)q_i^{SS}$$
$$=\frac{bZ(Y-\alpha)}{Z(n-1)(nZ+\beta)+b((n+1)Z+\beta)}\geqslant 0.$$

Therefore we can conclude that

$$\frac{\partial W^{SS}}{\partial d^{SO}}\geqslant 0. \quad \square$$

Note that we can easily show that

$$\hat{d}\geqslant \beta+\delta,$$

and therefore, if the fixed d_i is chosen equal to δ then $W^{SS}\geqslant W^{NZTS}$.

Proposition 9. *The social welfare of ISOSP-FD is maximized if the parameter d^{SO} is chosen equal to the threshold value $\hat{d}-\varepsilon$.*

Proof. We have established (Theorem 4) that $\partial W^{SS}/\partial d^{SO}\geqslant 0$, for $0\leqslant d^{SO}\leqslant \hat{d}-\varepsilon$. To prove this proposition, we demonstrate that for $d^{SO}>\hat{d}-\varepsilon$, we have $\partial W^{SS}/\partial d^{SO}\leqslant 0$. Under this condition, according to the equilibrium formulae, we have $b_i=\varepsilon$, and therefore, changing d^{SO} only modifies the equilibrium value of R_i (and not A_i and B_i). Therefore, we have

$$\frac{\partial W^{SS}}{\partial d^{SO}}=-\frac{1}{(\varepsilon+d^{SO})^2}\sum_{i,s}\left[\frac{\partial W^{SS}}{\partial y_{i,s}^{SS}}\frac{\partial y_{i,s}^{SS}}{\partial R_i}+\frac{\partial W^{SS}}{\partial x_{i,s}^{SS}}\frac{\partial x_{i,s}^{SS}}{\partial R_i}\right]. \tag{28}$$

Note that q_i is independent of R_i, and hence

$$\frac{\partial y_{i,s}^{SS}}{\partial R_i}=\frac{\partial x_{i,s}^{SS}}{\partial R_i}=\frac{Y_s-Y}{(1+nZR_i)^2}.$$

On the other hand, differentiating (26) yields

$$\frac{\partial W^{SS}}{\partial y_{i,s}^{SS}}=\theta_s(Y_s-\alpha-(Zn+\beta)y_{i,s}^{SS}),$$
$$\frac{\partial W^{SS}}{\partial x_{i,s}^{SS}}=\theta_s(-\delta x_{i,s}^{SS}).$$

Therefore, we conclude

$$\frac{\partial W^{SS}}{\partial d^{SO}}=-\frac{1}{(\varepsilon+d^{SO})^2}\sum_{i,s}\frac{\partial W^{SS}}{\partial y^{SS_{i,s}}}\frac{\partial y_{i,s}^{SS}}{\partial R_i}+\frac{\partial W^{SS}}{\partial x_{i,s}^{SS}}\frac{\partial x_{i,s}^{SS}}{\partial R_i}$$
$$=-\frac{1}{(\varepsilon+d^{SO})^2}\sum_{i,s}\frac{\theta_s(Y_s-Y)}{(1+nZR_i)^2}\cdot(Y_s-\alpha-(Zn+\beta)y_{i,s}^{SS}-\delta x_{i,s}^{SS})$$
$$=-\frac{1}{(\varepsilon+d^{SO})^2(1+nZR_i)^2}\sum_{i,s}\theta_s(Y_s-Y)\cdot(Y_s-\alpha-(Zn+\beta)y_{i,s}^{SS}-\delta x_{i,s}^{SS}).$$

To prove the theorem, we show $K_i = \sum_s \theta_s(Y_s - Y)\cdot(Y_s - \alpha - (Zn+\beta)y_{i,s}^{SS} - \delta x_{i,s}^{SS}) \geqslant 0$ for any i. Note that $y_{i,s}^{SS} = q_i^{SS} + x_{i,s}^{SS}$, and thus

$$\begin{aligned} K_i &= \sum_s \theta_s(Y_s - Y)(Y_s - \alpha - (Zn+\beta)y_{i,s}^{SS} - \delta x_{i,s}^{SS}) \\ &= \sum_s \theta_s Y_s^2 - Y^2 - (\alpha + Znq_i^{SS} + \beta q_i^{SS})\sum_s \theta_s(Y_s - Y) \\ &\quad - (Zn+\beta+\delta)\sum_s \theta_s(Y_s - Y)x_{i,s}^{SS}. \end{aligned}$$

Replacing $\sum_s \theta_s(Y_s - Y)$ with zero and inputting the value of $x_{i,s}^{SS} = ((Y_s - Y)R_i)/(1+nZR_i)$, we obtain

$$\begin{aligned} K_i &= \sum_s \theta_s Y_s^2 - Y^2 - (Zn+\beta+\delta)\sum_s \theta_s(Y_s - Y)\frac{(Y_s - Y)R_i}{1+nZR_i} \\ &= \left(\sum_s \theta_s Y_s^2 - Y^2\right)\left(1 - \frac{(Zn+\beta+\delta)R_i}{1+nZR_i}\right) \\ &= \left(\sum_s \theta_s Y_s^2 - Y^2\right)\left(\frac{1-(\beta+\delta)R_i}{1+nZR_i}\right). \end{aligned}$$

As discussed earlier, $\beta + \delta < \hat{d}$, and thus $0 < \beta + \delta < \hat{d} \leqslant d^{SO} < d^{SO} + \varepsilon$. If we multiply this by the positive quantity $R_i = 1/(d^{SO} + \varepsilon)$, we obtain $(\beta+\delta)R_i < 1$. Thus, we have

$$\frac{1-(\beta+\delta)R_i}{1+nZR_i} > 0.$$

Also, according to Jensen's inequality (or based on the nonnegativity property of variance), we can conclude $\operatorname{var}(Y_s) = \sum_s \theta_s Y_s^2 - Y^2 \geqslant 0$. These inequalities indicate that $K_i \geqslant 0$, and therefore, we can conclude

$$\frac{\partial W^{SS}}{\partial d^{SO}} = -\frac{1}{(\varepsilon + d_i)^2(1+nZR_i)^2}\sum_i K_i \leqslant 0. \quad \square$$

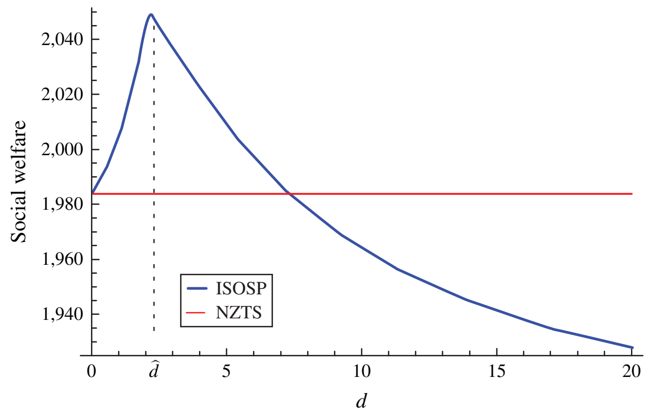
Example 1. Consider a market with two symmetric generators as defined in Table 1.

Figure 2 shows how the social welfare of the stochastic settlement mechanism is affected by the choice of d^{SO}. It also demonstrates that for $d^{SO} < \hat{d}$ and even beyond, the stochastic settlement mechanism has a higher equilibrium social welfare in comparison with the NZTS mechanism. Note that at $d^{SO} = \hat{d}$, the equilibrium b_i is set to ε and social welfare is maximized. For the rest of this example, we assume that the ISO chooses $d^{SO} = 0.5$, which ensures higher equilibrium social welfare from the stochastic settlement in comparison with the NZTS mechanism.

Table 1. The Market Environment for the Example

Parameter	Value
α, β, δ	50, 1, 0.5
Y_1, Y_2, Z	100, 150, 1
θ_1, θ_2	0.5, 0.5
n	2

Figure 2. (Color online) The Effect of d^{SO} on the Social Welfare of the Stochastic Settlement Model and on How It Compares to the Two Period Settlement Mechanism



Note. The thicker line traces ISOSP-FD welfare whereas the thinner line traces the NZTS welfare.

Another interesting experiment is to investigate the effect of β and δ on these mechanisms.

Figures 3–5 compare the ISOSP-FD and the NZTS mechanisms for this example, however for different β and δ values. A first observation is that the stochastic settlement mechanism increases social and consumer welfare and decreases producer welfare in comparison with the two settlement mechanism.

It is also interesting to investigate the effect of competition on these mechanisms. To do so, we can test the effect of number of firms on these mechanism.

Figure 6 shows the difference in the social welfare of our two mechanisms as a function of n. It shows that when the number of generators increases, the performance of the stochastic and two-period settlement mechanisms converge.

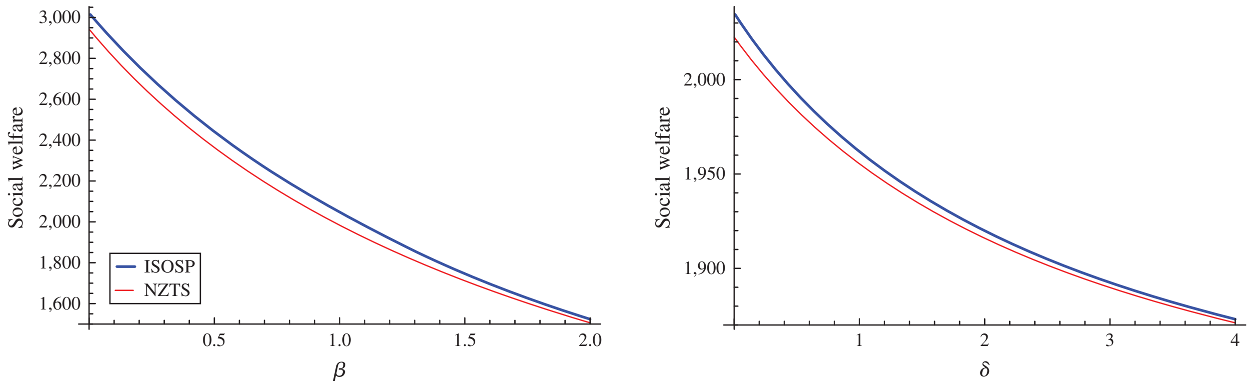
6. Robustness to Modelling Assumptions

In this section, we investigate the robustness of our results to two important model assumptions.

6.1. The Case of Asymmetric Generators

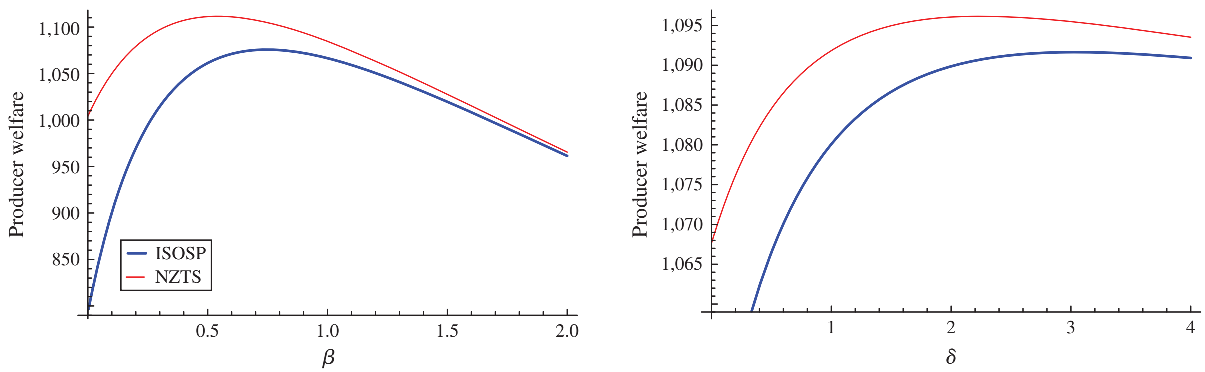
Thus far, we have derived the analytical expressions for a symmetric equilibrium. We have also proved that the stochastic settlement market always improves social welfare under this symmetric equilibrium. We now consider a duopoly with two asymmetric generators. As outlined in the introduction of this paper, there are a number of papers that are relevant to the work presented in this paper. However, none of the existing papers deals with the short-term penalty costs in a two-period market clearing paradigm. Furthermore, the simplification techniques that we have employed to derive analytical expressions for our symmetric case no

Figure 3. (Color online) Social Welfare of ISOSP-FD and NZTS for Different β and δ Values



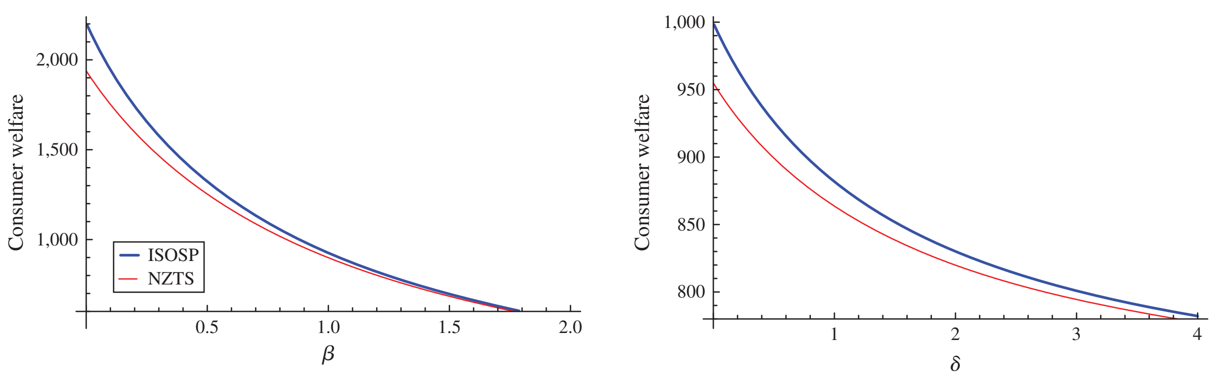
Note. The thicker lines trace ISOSP-FD welfare while the thin lines trace the NZTS welfare.

Figure 4. (Color online) Producer Welfare of ISOSP-FD and NZTS for Different β and δ Values



Note. The thick lines trace ISOSP-FD producer welfare while the thin lines trace NZTS producer welfare.

Figure 5. (Color online) Consumer Welfare of ISOSP-FD and NZTS for Different β and δ Values



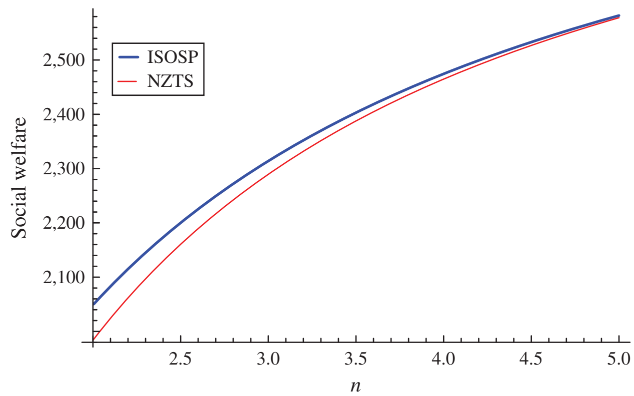
Note. The thick lines trace ISOSP-FD consumer welfare while the thin lines trace NZTS consumer welfare.

longer apply here because of assumption of asymmetry. At this point, we use a computational method laid out in Section 6.1.3 to construct an equilibrium of our asymmetric game. In what follows, we will remind the reader of market assumptions, then restate the ISO's market clearing problem before proceeding to equilibrium computations.

6.1.1. The Market Environment in the Asymmetric Case. This environment is very similar to the setup for the symmetric case. It has the following distinguishing features.

- The generators are no longer symmetric. To make the examples computationally tractable, we focus on two generator examples.
- We use the same form of long-run and short-run cost functions as the symmetric case. Here however, the firms are no longer identical and the cost parameters α_i, β_i, and δ_i depend on the firm i.

Figure 6. (Color online) Social Welfare of the Two-Period Settlement Mechanism Converges to That of the Stochastic Mechanism When n Increases



Notes. Competition increases with a bigger market. The thick line traces ISOSP-FD welfare while the thin line traces NZTS welfare.

6.1.2. Models. The ISO's optimization problem is similar to the symmetric case, however Theorem 8 no longer applies in the asymmetric case. Therefore to ensure the nonnegativity of the equilibrium, both for the two-period settlement and the stochastic programming market clearing mechanisms, we need to enforce nonnegativity in the ISO's problem. Thus, the predispatch problem of the ISO in the (asymmetric) NZTS mechanism is

$$\begin{aligned}\text{PDATS:}\quad &\min_{q,Q}\ z,\quad z=\sum_{i=1}^{n}\left(a_i q_i+\frac{b_i}{2}q_i^2\right)-\left(YQ-\frac{Z}{2}Q^2\right)\\ &\text{s.t.}\ \sum_i q_i-Q=0,\quad [f]\\ &\qquad q_i\geqslant 0,\quad \forall\, i.\end{aligned}\tag{29}$$

The ISO's spot market optimization problem for scenario s is

$$\begin{aligned}\text{SATS(s):}\quad &\min_{y_s,C_s}\ z,\quad z=\sum_{i=1}^{n}\left(a_i y_{i,s}+\frac{b_i}{2}y_{i,s}^2\right)-\left(Y_sC_s-\frac{Z}{2}C_s^2\right)\\ &\text{s.t.}\ \sum_i y_{i,s}-C_s=0,\quad [p_s]\\ &\qquad y_{i,s}\geqslant 0,\quad \forall\, i.\end{aligned}\tag{30}$$

Similarly, the stochastic optimization problem of the ISO in the stochastic settlement mechanism can be represented as

$$\begin{aligned}\text{SATS:}\quad \min\ z,\quad z=&\sum_{s=1}^{S}\theta_s\left(\sum_{i=1}^{n}\left[a_i(q_i+x_{i,s})+\frac{b_i}{2}(q_i+x_{i,s})^2\right.\right.\\ &\left.\left.+\frac{d_i}{2}x_{i,s}^2\right]-\left(Y_sC_s-\frac{Z}{2}C_s^2\right)\right)\\ \text{s.t.}\ &\sum_i q_i-Q=0,\quad [f]\\ &Q+\sum_i x_{i,s}-C_s=0,\quad \forall s,\quad [p_s]\\ &q_i+x_{i,s}\geqslant 0,\quad \forall i,s.\end{aligned}$$

6.1.3. Equilibrium Computations. To find a Nash equilibrium to our games, we use a dynamic process. The idea is to allow each participant in turn to update its strategies, assuming the strategy set of the other participants is fixed. If this diagonalization process terminates with no participant willing to deviate from its last strategy, then we have arrived at a Nash equilibrium. Note that by embedding the optimality conditions of the market clearing problems (for the NZTS, or the stochastic programming market clearing), the generator optimization problem becomes nonconvex and requires solving to global optimality. To do this, we have used the global solver of LINGO. The global solver of LINGO guarantees the optimality of its final solution using a branch-and-bound approach. Here a sequence of piecewise convex relaxations of the original (nonconvex) problem are solved. The convex relaxations are derived using bounds on the variables. If the optimal solution of the relaxed problem is feasible for the original problem, it is also the optimal point of the original problem. If not, further enhancement is made through dividing up the domain of the objective function and creating more accurate, piecewise convex functions on each part of the domain. The process of branching continues until all branches end with an optimal point. Note that user defined tolerances on the splitting procedure make this method a finite process. For more information about the mathematics behind this global solver, see Lin and Schrage (2009). The tolerance that we have used, as the minimum acceptable difference between best response strategies of firms in different turns, is of order of 10^{-10}.

6.1.4. Sensitivity to Different Cost Structures (i.e., Generation Technologies). Different generation technologies have different structure in their cost functions, e.g., a particular generation technology may have a high generation cost but a low cost for fast deviation and another generator might be the opposite. In this section, we analyze a market with two asymmetric generators with various cost patterns (different layouts of α_i, β_i, and, δ_i). Without loss of generality we call the generator with the lower α_i value generator 1 and the other generator 2. We then design different experiments with different possibilities for β_i and δ_i (e.g., $\beta_1>\beta_2$, $\beta_1<\beta_2$, etc.). For each of these layouts, we consider two cases for d^{SO}: $d^{\text{SO}}=\min_i\{\delta_i\}$ and $d^{\text{SO}}=\max_i\{\delta_i\}$. Consider a market with two demand scenarios with parameters given in Table 2. Table 3 summarizes the difference between the equilibrium values of the SFSP-FD and NZTS mechanisms for each of these experiments. According to these results, SFSP-FD results in lower prices, profit, and producer welfare,

Table 2. Scenarios and Demand

Scenarios	θ	Y
1	0.5	100
2	0.5	150

Table 3. Difference in the Equilibrium Values of the SFSP-FD and NZTS (i.e., SFSP-FD – NZTS) Under Asymmetry Assumption and Different Cost Patterns

Ex.	Gen	d^{SO}	f	$p(s_1)$	$p(s_2)$	Profit	CW	PW	SW
1	1	0.001	−0.0057	−0.01	−0.01	−0.19	0.48	−0.30	0.18
	2					−0.11			
2	1	0.001	−0.014	−0.01	−0.02	−0.58	1.22	−0.81	0.41
	2					−0.23			
3	1	0.5	−2.58839	−2.59	−2.59	−67.40	218.43	−119.41	99.02
	2					−52.00			
4	1	0.001	−0.00775	−0.01	−0.01	−0.23	0.64	−0.37	0.27
	2					−0.14			
5	1	0.001	−0.02637	−0.03	−0.03	−0.95	2.16	−1.05	1.10
	2					−0.10			
6	1	0.5	−2.4747	−2.48	−2.47	−58.07	202.06	−140.63	61.44
	2					−82.56			
7	1	0.5	−3.55014	−3.55	−3.55	−160.97	305.33	−231.03	74.30
	2					−70.07			
8	1	0.5	−3.99094	−3.99	−3.99	−91.25	348.13	−174.46	173.67
	2					−83.21			
9	1	0.5	−3.6365	−3.64	−3.64	−97.94	305.61	−200.01	105.60
	2					−102.07			
10	1	0.5	−3.72095	−3.72	−3.72	−89.78	311.02	−282.17	28.86
	2					−192.38			

and higher consumer and social welfare in comparison with the NZTS mechanism.

6.1.5. Sensitivity to d^{SO}. To analyze the sensitivity of our results to the value of d^{SO}, we focus on the experiment in Section 6.1.4 with the cost parameters listed in Table 4. To compare the stochastic settlement mechanism with the NZTS, we find the equilibrium values of the SFSP-FD mechanism on a range of different values for d^{SO}. The equilibrium values of the two-period single settlement mechanism and the stochastic settlement mechanism for different bs are listed in Table 5.

This table indicates that our proven results of the symmetric case are expected in this case as well. Firstly, SFSP-FD yields higher social welfare for $d^{SO} \in (0, \max_i\{\delta_i\})$. Secondly, social welfare is increasing with respect to d^{SO} in this range and reaches its maximum value at a much higher level of d^{SO} (somewhere between 2 and 3 in this example). After this point, social welfare starts to drop with higher d^{SO} values and ends up lower than that of the NZTS mechanism for very large values of d^{SO}. The third similarity is that generators submit $b = \varepsilon$ when d^{SO} is larger than a threshold value.

6.2. Restriction to the Case of Supply Functions with Intercept Zero

The model originally used by Green (1996) restricted the linear supply offers to have an intercept at zero.

Table 4. Cost Function of the Generators

Gen	α	β	δ
1	0	0.001	0.001
2	10	1	0.5

We went through the exercise of constructing NZTS and ISOSP equilibrium results when supply functions comply with the zero intercept rule (note that this eliminates one variable from the decision space of the generators). For this case, we restricted the model to a duopoly. The methodology we have used for this case follows that of the general case. For each market clearing mechanism we establish the values of q_i, f, $x_{i,s}$, and p_s as before. Then we obtain the expected utility expressions and establish quasiconcavity results for each case following the same lines of argument used in the previous sections.

When restricted by the zero intercept condition, and responding to the opponent restricted by the same condition, the best reply and hence equilibrium values change. Furthermore, we find that in the ISOSP case, we have a unique (symmetric) equilibrium. An interesting observation here is that allowing one more degree of freedom, by the choice of an intercept, leading to a continuum of equilibria, leads the ISO to acknowledge a deviation cost (penalty) for the participants (in efforts to improve welfare). As we observed, for any choice of deviation penalty in $(0, \delta)$, the welfare of ISOSP is improved over NZTS. This is no longer the case for the Green type linear supply functions. That is, the welfare difference between NZTS and ISOSP can be positive or negative. Specifically, if we fix $\beta = 2.0$, $\delta = 4.0$, and note $W^{NZTS} - W^{SS}$ by TWD, we find the following.

$Y = 5$	$Y = 1.25$
$\sigma_Y^2 = 0.25$	$\sigma_Y^2 = 0.0625$
$Z = 0.5$	$Z = 0.125$
$TWD = -4.26892$	$TWD = 1.54577$

Table 5. Equilibrium Values of SFSP-FD for Different Levels of d^{SO} in Comparison with the Equilibrium Values of NZTS

Gen	d^{SO}	a	b	$b+d$	CW	PW	SW
1	0.0001	−0.0569	0.6563	0.6564	3,688.9	3,092.4	6,781.3
2	0.0001	−1.0159	1.8962	1.8963			
1	0.001	−0.0078	0.6554	0.6564	3,689.3	3,092.1	6,781.4
2	0.001	−1.0042	1.8953	1.8963			
1	0.01	0.4858	0.6464	0.6564	3,693.3	3,089.5	6,782.8
2	0.01	−0.8876	1.8863	1.8963			
1	0.1	5.4164	0.5565	0.6565	3,735.6	3,060.9	6,796.5
2	0.1	0.2343	1.7963	1.8963			
1	0.5	27.2341	0.1566	0.6566	3,993.3	2,861.7	6,855.0
2	0.5	3.7615	1.3955	1.8955			
1	2	15.8928	ε	2	6,027.3	1,782.8	7,810.0
2	2	14.9124	0.1667	2.1667			
1	3	15.7103	ε	3	6,022.1	1,769.4	7,791.5
2	3	15.7103	ε	3			
1	10	36.2862	ε	10	3,943.7	2,692.3	6,636.1
2	10	36.2862	ε	10			
1	100	36.2862	ε	100	3,935.2	2,620.8	6,556.0
2	100	36.2862	ε	100			
1	NZTS	−0.0622	0.6567	NA	3,688.0	3,092.7	6,780.7
2	NZTS	−1.0199	1.8964	NA			

7. Conclusion

In this paper, we set up a simple modelling environment in which we were able to compare the New Zealand inspired two-period single settlement market clearing mechanism against a stochastic settlement auction, which reduces to another two-period single settlement auction with explicit penalties of deviation, therefore different from the NZTS model. We were able to model firms' best responses in these markets, and so find equilibrium behaviour in each. We find that in our symmetric models, the ISOSP-FD auction provably dominates the NZTS auction when measuring expected social welfare.

To test the robustness of our results to the modeling assumptions, we extended our analysis to two other cases. For the case of asymmetric generators, ISOSP-FD outperforms NZTS (assuming that the deviation parameter is chosen appropriately). However, the same results cannot be replicated for a restricted model where affine offers are confined to have intercept zero, and we provide a case for which NZTS has a better performance.

Endnotes

[1] Insofar as using these estimates to make an offer, the information is only useful up to gate closure as thereafter offers can not be changed. There is no capacity market in the NZEM, therefore it is necessary to lock in generation offers well ahead of time to avoid a situation where demand must be curtailed involuntarily.

[2] The current financial settlement in the NZEM is based on ex post prices that are computed with average demand over a period. Constrained on and off payments are used to ensure sufficient payment is made to the generators. However the Electricity Authority is now considering real-time pricing and is in consultation with the stakeholders.

[3] Within this five-minute period a frequency keeping generator will match any small changes in demand. We ignore this aspect of the market, as frequency keeping is purchased through a separate market and until recently was procured through annual contracts.

[4] While it may be tidier from an exposition perspective to allow for tranche offers (with fixed price and maximum quantity), such a model would lead to maximum generation constraints, and their dual multipliers, in the ISO's problem. This will in turn impede obtaining an analytical equilibrium expression.

[5] This is a modified version of Pritchard et al. problem. There is only one node and thus no transmission constraints, and demand is elastic.

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Technical Companion for the Paper
"Single and Multi-settlement Approaches to Market Clearing under Demand Uncertainty"

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1 Proposition 4

Proposition. *The equilibrium pre-dispatch and spot production quantities of the firms in the NZTS market are non-negative, i.e.* $q_i \geq 0 \quad \forall i$, *and* $y_{i,s} \geq 0 \quad \forall i, s$ *where* q_i *and* $y_{i,s}$ *are the optimal solutions to problems (1) and (2) respectively using the equilibrium parameters from (10) and (11) from the main paper.*

Proof. To prove the proposition, we first show the equilibrium price intercept of the supply function of generators (i.e. $a_i = \frac{A_i}{B_i}$) is less than the price intercept of the demand function (i.e. Y and Y_s). Then we show this property entails the non-negativity of equilibrium quantities.

Substituting A_i and B_i from Proposition 3 into $a_i = \frac{A_i}{B_i}$, and then taking the derivative of a_i with respect to Z, we achieve

$$\frac{\partial a_i}{\partial Z} = \frac{2\delta\left((n-2)^2 Z + 2k + n(\beta+\delta+k)\right)(Y-\alpha)}{k((n+2)Z+\beta-\delta+k)^2},$$

where $k = \sqrt{(n-2)^2 Z^2 + 2nZ(\beta+\delta) + (\beta+\delta)^2}$. Because $n \geq 2$, $Z > 0$, $\beta \geq 0$, $\delta \geq 0$, and $\alpha \leq Y$, we have

$$\frac{\partial a_i}{\partial Z} \geq 0. \tag{1}$$

On the other hand, taking the limit of a_i as Z approaches infinity, we obtain

$$\lim_{Z\to\infty} a_i = \alpha. \tag{2}$$

Equations (1) and (2) yield

$$a_i \leq \alpha.$$

This together with assumption $\alpha \leq Y_s$,$\forall s$ yields

$$a_i \leq Y_s \qquad \forall i, s. \tag{3}$$

Using $a_i = \frac{A_i}{B_i}$, we can rewrite equation (3) as

$$B_i Y_s - A_i \geq 0 \qquad \forall i, s. \tag{4}$$

Also, using the value of B_i from Proposition 3, we can show $B_i \geq 0$. Thus, we can conclude

$$B \geq 0. \tag{5}$$

On the other hand, embedding p_s into $y_{i,s}$ from Proposition 2, we obtain

$$y_{i,s} = \frac{B_i Y_s - A_i}{ZB + 1} \qquad \forall i, s.$$

This together with equations (4) and (5) and assumption $Z > 0$ gives

$$y_{i,s} \geq 0 \qquad \forall i, s.$$

From Propositions 1 and 2, we achieve $q_i = \sum_s \theta_s y_{i,s}$. As $\theta_s \geq 0$, we obtain

$$q_i \geq 0 \qquad \forall i.$$

□

2 The optimal solution to ISOSP problem: proof of Proposition 5

Proposition. *If* $(\mathbf{q}, \mathbf{x}, \mathbf{f}, \mathbf{p})$ *represents the solution of ISOSP, then we have*

$$\begin{aligned} q_i &= \frac{(Y + ZA)B_i}{1 + ZB} - A_i && (6)\\ x_{i,s} &= \frac{(Y_s - Y)R_i}{1 + ZR} && (7)\\ f &= \frac{Y + ZA}{1 + ZB} \\ p_s &= \frac{Y + ZA}{1 + ZB} + \frac{Y_s - Y}{1 + ZR}. \end{aligned}$$

Proof. The Lagrangian function of ISOSP can be represented as follows:

$$\begin{aligned} L &= -f\left(-Q + \sum_{i=1}^{n} q_i\right) \\ &\quad + \sum_{s=1}^{S} \theta_s \left(-p_s \left(Q - C_s + \sum_{i=1}^{n} x_{i,s} \right) \right. \\ &\quad \left. -Y_s C_s + \frac{Z C_s^2}{2} + \sum_{i=1}^{n} \left(\frac{1}{2} d_i x_{i,s}^2 + a_i \left(q_i + x_{i,s} \right) + \frac{1}{2} b_i \left(q_i + x_{i,s} \right)^2 \right) \right). \end{aligned}$$

Taking the derivative with respect to different variables yields the following equations.

$$\frac{\partial L}{\partial q_i} = -f + \sum_s \theta_s \left(a_i + b_i \left(q_i + x_{i,s}\right)\right) \tag{8}$$

$$\frac{\partial L}{\partial x_{i,s}} = \theta_s \left(-p_s + a_i + b_i \left(q_i + x_{i,s}\right) + d_i x_{i,s}\right) \tag{9}$$

$$\frac{\partial L}{\partial C_s} = \theta_s \left(p_s - Y_s + ZC_s\right) \tag{10}$$

$$\frac{\partial L}{\partial Q} = f - \sum_s \theta_s p_s \tag{11}$$

$$\frac{\partial L}{\partial p_s} = \theta_s \left(-Q + C_s - \sum_i x_{i,s}\right) \tag{12}$$

$$\frac{\partial L}{\partial f} = Q - \sum_i q_i \tag{13}$$

The Lagrangian is evidently a convex function. Thus, for finding the solution of the stochastic program, we should set all above derivatives to zero. From (8)

$$f = a_i + b_i q_i + \sum_s p_s x_{i,s}. \tag{14}$$

From (9) and (14)

$$p_s = f + (b_i + d_i) x_{i,s}, \tag{15}$$

and from (11)

$$f = \sum_s \theta_s p_s. \tag{16}$$

Now (14), (15) and (16) result in the following conclusion, as it is also concluded from Lemma 2.

$$\sum_s \theta_s x_{i,s} = 0. \tag{17}$$

Equations (14) and (17) lead to

$$f = a_i + b_i q_i. \tag{18}$$

Note that the forward price is solely a function of forward quantities.

From (10),

$$p_s = Y_s - ZC_s, \tag{19}$$

from (12),

$$C_s = Q + \sum_i x_{i,s}, \tag{20}$$

and from (13),

$$Q = \sum_i q_i. \tag{21}$$

Equations (17) and (20) lead to

$$\sum_s \theta_s C_s = Q. \tag{22}$$

Equations (16), (19) and (22) lead to

$$f = Y - ZQ. \tag{23}$$

Now from (18) and (23) we can conclude

$$q_i = \frac{Y - ZQ - a_i}{b_i}. \tag{24}$$

In consequence, using (21), summing q_i from (24) over all firms and applying the change of variables from (a_i, b_i, d_i) to (A_i, B_i, R_i), we obtain

$$Q = (Y - ZQ)B - A.$$

Therefore,

$$Q = \frac{YB - A}{1 + ZB}. \tag{25}$$

Now the following inference can be resulted from (24) and (25).

$$q_i = \frac{(Y + ZA)B_i}{1 + ZB} - A_i. \tag{26}$$

Now let us find $x_{i,s}$. Equations (15), (19) and (20) give

$$f + (b_i + d_i)x_{i,s} = Y_s - ZQ - Z\sum_i x_{i,s}.$$

By adding (23) to this equation we obtain

$$x_{i,s} = \frac{Y_s - Y - Z\sum_i x_{i,s}}{b_i + d_i}. \tag{27}$$

Now by getting a summation from (27) and simplifying the resulting equation we achieve

$$\sum_i x_{i,s} = \frac{(Y_s - Y)R}{1 + ZR}.$$

By inserting this equation in (27), we obtain

$$x_{i,s} = \frac{(Y_s - Y)R_i}{1 + ZR}, \tag{28}$$

and from (23) and (25), the first stage price can be extracted:

$$f = \frac{Y + ZA}{1 + ZB}. \tag{29}$$

One observation about this equation is that forward price is independent of R, in other words, it is independent of the deviating cost in the spot market.

Equations (25), (28) and (29) determine the spot price for each scenario:

$$p_s = \frac{Y + ZA}{1 + ZB} + \frac{Y_s - Y}{1 + ZR}. \tag{30}$$

□

3 The equilibrium of the stochastic settlement market: proof of Proposition 7

Proposition. *The unique symmetric equilibrium quantities of ISOSP-FD are as follows.*

$$b_i = \max\{\varepsilon, \frac{-Z(n-2) + \beta + \delta + \sqrt{Z^2(n-2)^2 + 2Zn(\beta+\delta) + (\beta+\delta)^2}}{2} - d^{SO}\} \tag{31}$$

$$a_i = \frac{\alpha - Y + B_i\left(-Z(Y(n-2) - (2n-1)\alpha) + Y\beta + Z(n-1)(Zn\alpha + Y\beta)B_i\right)}{B_i\left(Z(n+1) + \beta + Y(n-1)(Zn+\beta)B_i\right)} \tag{32}$$

Proof. To find a symmetric equilibrium, we can use

$$B_{-i} = (n-1)B_i,$$

$$A_{-i} = (n-1)A_i,$$

and

$$R_{-i} = (n-1)R_i.$$

By putting these equations in the best response functions (from Theorem 1) and solving the resulting equations with respect to A_i and R_i, we obtain the following equilibrium equations.

$$A_i = \frac{\alpha - Y + B_i\left(-Z(Y(n-2) - (2n-1)\alpha) + Y\beta + Z(n-1)(Zn\alpha + Y\beta)B_i\right)}{Z(n+1) + \beta + Y(n-1)(Zn+\beta)B_i}$$

$$R_i = \min\{B_i, \frac{2}{-Z(n-2) + \beta + \delta + \sqrt{Z^2(n-2)^2 + 2Zn(\beta+\delta) + (\beta+\delta)^2}}\}$$

Let us see why equation (31) implies a true equilibrium quantity. Let

$$\hat{R}_i = \frac{2}{-Z(n-2)+\beta+\delta+\sqrt{Z^2(n-2)^2+2Zn(\beta+\delta)+(\beta+\delta)^2}}.$$

If $\hat{R}_i \le B_i$, it satisfies the best response function for R_i. When $\hat{R}_i > B_i$, we need to show

$$\frac{1+Z(n-1)B_i}{Z+\beta+\delta+Z(n-1)(\beta+\delta)B_i} \ge B_i.$$

It means when the other generators j have chosen $R_j = B_j$, the best response for the firm i is also to choose $R_i = B_i$. Note that B_i is a fixed quantity chosen by the ISO, Thus, $B_j = B_i$. Now define

$$f(x) = \frac{1+Z(n-1)x}{Z+\beta+\delta+Z(n-1)(\beta+\delta)x} - x.$$

We can easily show that $f(x)$ is a concave function for $x \ge 0$:

$$f^{''}(x) = -\frac{2Z^3(n-1)^2(\beta+\delta)}{(Z+\beta+\delta+Z(n-1)(\beta+\delta)x)^3} < 0.$$

Also $f(0) = \frac{1}{Z+\beta+\delta} > 0$ and $f(\hat{R}_i) = 0$. Thus for $0 < B_i < \hat{R}_i$, and by considering concavity of $f(x)$, we obtain $f(B_i) \ge 0$. Therefore

$$\frac{1+Z(n-1)B_i}{Z+\beta+\delta+Z(n-1)(\beta+\delta)B_i} \ge B_i.$$

□

4 Stochastic settlement yields non-negative equilibria: proof of Proposition 8

Proposition. *Let* $(\mathbf{q}^*, \mathbf{x}^*)$ *represent an equilibrium of ISOSP-FD. Then the following inequalities hold.*

$$\forall i, s : \; q_i^* + x_{i,s}^* \ge 0$$

$$\forall i : \; q_i^* \ge 0$$

Proof. From (26) and (28), the following equation can be derived:

$$y_{i,s} = q_i^* + x_{i,s}^* = \frac{(Y+ZA)B_i}{1+ZB} - A_i + \frac{(Y_s - Y)R_i}{(1+ZR)}.$$

It is obvious that if $y_{i,s}$ is non-negative for the scenario that has the lowest Y_s, it is non-negative for the other scenarios as well. Thus, we prove this only for the scenario s' for which we have $Y_{s'} \leq Y_s$ for all s. If we assume having at least two different scenarios with positive probabilities, we have

$$Y_{s'} < Y. \tag{33}$$

Let us first define

$$\hat{R}_i = \frac{2}{-Z(n-2) + \beta + \delta + \sqrt{Z^2(n-2)^2 + 2Zn(\beta+\delta) + (\beta+\delta)^2}}.$$

Now consider $y^{'}_{i,s'} = \min_{\alpha,\delta} y_{i,s'}$. Obviously if we prove that $y^{'}_{i,s'}$ is non-negative, we have also proven the non-negativity of $y_{i,s}$. Now $y_{i,s}$ satisfies

$$\frac{\partial y_{i,s'}}{\partial \delta} = \begin{cases} \text{if } \hat{R}_i \leq B_i : \\ \frac{2\left(Zn+\beta+\delta+\sqrt{Z^2(n-2)^2+2Zn(\beta+\delta)+(\beta+\delta)^2}\right)(Y-Y_s)}{\sqrt{Z^2(n-2)^2+2Zn(\beta+\delta)+(\beta+\delta)^2}\left(Z(n+2)+\beta+\delta+\sqrt{Z^2(n-2)^2+2Zn(\beta+\delta)+(\beta+\delta)^2}\right)^2} \\ \\ \text{Otherwise:} \\ 0. \end{cases}$$

$$\frac{\partial y_{i,s'}}{\partial \alpha} = -\frac{1 + ZB_1(n-1)}{Z(n+1) + \beta + ZB_1(n-1)(Zn+\beta)}.$$

The parameters Z, β, and δ are non-negative. Thus from (33) we can conclude

$$\frac{\partial y_{i,s}}{\partial \delta} \geq 0,$$

$$\frac{\partial y_{i,s}}{\partial \alpha} \leq 0.$$

Consequently, $\delta = 0$ and $\alpha = Y_{s'}$ minimize $y_{i,s'}$. Note that we have assumed that the y-intercept of the cost function α is less than the y-intercept of the demand scenarios $Y_{s'}$. Thus, we prove that $y^{'}_{i,s'} = y_{i,s'}(\delta = 0, \alpha = Y_{s'})$ takes non-negative values.

When $\delta = 0$, at $\hat{\beta} = \frac{1+ZB_i(n-2)}{B_i(1+ZB_i(n-1))}$, we have $\hat{R}_i = B_i$. By applying the fact that $\hat{R}_i$ is a decreasing function of β, we can conclude

$$R_i = \begin{cases} B_i & \beta < \hat{\beta} \\ \hat{R}_i & \beta \geq \hat{\beta} \end{cases}$$

and

$$y^{'}_{i,s'} = \begin{cases} \frac{(Y+ZA)B_i}{1+ZB} - A_i + \frac{(Y_s-Y)B_i}{(1+ZB)} & \text{if } \beta < \hat{\beta} \\ \frac{(Y+ZA)B_i}{1+ZB} - A_i + \frac{(Y_s-Y)\hat{R}_i}{(1+Z\hat{R})} & \text{if } \beta \geq \hat{\beta}. \end{cases}$$

We can also show that equation $y'_{i,s'} = 0$ only holds at $\beta = \hat{\beta}$. In addition, $y'_{i,s'}$ is a continuous function. These mean $y'_{i,s'}$ is either entirely positive or entirely negative in each of $[0, \hat{\beta}]$ or $[\hat{\beta}, \infty)$. Firstly, we prove that it is positive in $[0, \hat{\beta}]$.

We see that $\frac{\partial y'_{i,s'}}{\partial A_i} < 0$. On the other hand,

$$\frac{\partial A_i}{\partial \beta} = \frac{(Y-\alpha)\left(1+Z(n-1)B_i\right)^2(1+ZnB_i)}{\left(Z(n+1)+\delta+Z(n-1)(Zn+\beta)B_i\right)^2} \geq 0.$$

Therefore, for $\beta < \hat{\beta}$, $\frac{\partial y'_{i,s'}}{\partial \beta} = \frac{\partial y'_{i,s'}}{\partial A_i}\frac{\partial A_i}{\partial \beta}$ is not positive. It means $y_{i,s'}$ is a non-increasing function of β in this interval. Considering the fact that $y'_{i,s'}(\hat{\beta}) = 0$, we can conclude

$$y'_{i,s'} \geq 0 \text{ if } \beta \leq \hat{\beta}. \tag{34}$$

The right derivative of $y'_{i,s'}$ at $\hat{\beta}$ also has a positive value of

$$\frac{Z^2\left(Y-Y_s\right)B_i(n-1)\left(1+Z(n-1)B_i\right)\left(1+ZnB_i\right)^2}{\sqrt{Z^2(n-2)^2+2Zn\beta+\delta^2}\left(Z(n+1)+\delta+ZB_i\left(-\beta+2n(Zn+\beta)+Z(n-1)n(Zn+\beta)B_i\right)\right)^2}.$$

If we add this to the facts that $y'_{i,s'}(\hat{T}) = 0$ and $y'_{i,s'}$ is either entirely non-negative or entirely non-positive for $\beta > \hat{\beta}$, we can conclude that

$$y'_{i,s'} \geq 0, \text{ if } \beta \geq \hat{\beta}. \tag{35}$$

Equations (34) and (35) can be gathered to conclude

$$y'_{i,s'} \geq 0.$$

Therefore

$$y_{i,s} = q_i^* + x_{i,s}^* \geq 0.$$

We know from Lemma 2 that $x_{i,s}^*$ is non-positive for at least one-scenario. Thus $q_i^* \geq 0$.

□

5 Equilibrium of the stochastic settlement mechanism with non-negativity constraints: Theorem 3

To prove Theorem 3, we consider three different but closely related optimization problems (for a specified firm). These are:

1. WONN: The optimization problem of a generator assuming that the non-negativity constraints are not contained in the market clearing problem. The optimal solution to this problem (assuming that the other generators offer the equilibrium values for ISOSP, given in the paper) is the equilibrium offer strategy for this firm under ISOSP.

2. WNN: The optimization problem of a generator assuming that the non-negativity constraints are imposed for the market clearing problem.

3. RWNN: This is a relaxed version of WNN. We remove the orthogonality constraints $e_{i,s}(q_i+x_{i,s}) = 0$ for any firm i, and non-negativity constraints $q_i + x_{i,s} \geq 0$ for all non-optimizing firms $(i \neq j)$.

We wish to establish that the optimal solution to WONN remains optimal for WNN. The feasible sets of these problems contain the optimal solutions of the ISO's problem. Since ISO's problem is different with and without non-negativity constraints, there are points (i.e. a vector of decisions variables such as offers, prices, dispatch quantities) feasible only for WNN, and points feasible only for WONN. Therefore, if we show that the optimal solution to WONN is also feasible in WNN, it does not suffice to prove the theorem, since there are other feasible solutions of WNN that are not investigated in WONN. This makes it difficult to present an intuitive proof for the theorem. To prove the theorem, we use Lemma 5.3 to show that the optimal solution to WONN is also the optimal solution to RWNN. Then, in the proof of Theorem 3, we establish that this point is also a feasible solution to WNN. Since it is optimal for the relaxed problem and it is feasible in the more constrained WNN problem, this point is also the optimal solution to WNN. Therefore, it is also the equilibrium point of ISOSP with non-negativity constraints. Lemmas 5.1 and 5.2 are technical lemmas that help establish the main lemma, Lemma 5.3. Lemma 5.3 is the only lemma that is directly used in the proof of Theorem 3.

5.1 ISOSP clearing problem with non-negativity constraints

The SP clearing problem with non-negativity constraints is

ISOSP :

$$
\begin{aligned}
\min z \quad &= \quad \sum_{s=1}^{S} \theta_s \left(\sum_{i=1}^{n} \left[a_i(q_i + x_{i,s}) + \frac{b}{2}(q_i + x_{i,s})^2 + \frac{d_i}{2} x_{i,s}^2 \right] - (Y_s C_s - \frac{Z}{2} C_s^2) \right) \\
s.t. \quad & \quad \sum_i q_i - Q = 0 \\
& \quad Q + \sum_i x_{i,s} - C_s = 0 \quad \forall s \in \{1, \dots, S\} \\
& \quad q_i + x_{i,s} \geq 0 \quad \forall i \in \{1, \dots, n\}, \forall s \in \{1, \dots, S\}
\end{aligned}
$$

ISOSP is a convex optimization problem as the objective function of ISOSP is a convex function, and its constraints are linear. Therefore, solving the KKT conditions of this problem is equivalent to solving ISOSP.

5.1.1 KKT of ISOSP

To find the KKT conditions we can use the Lagrangian function

$$
\begin{aligned}
L = \sum_{s=1}^{S} \Bigg(\theta_s \Bigg(\sum_{i=1}^{n} \Bigg(a_i \left(x_{i,s} + q_i\right) + \frac{b}{2} \left(x_{i,s} + q_i\right)^{2} + \frac{d_i}{2} x_{i,s}^2 \Bigg) \\
- \left(C_s Y_s - \frac{Z C_s^2}{2} \right) + p_s \left(Q + \sum_{i=1}^{n} x_{i,s} - C_s \right) \Bigg) \\
- \sum_{i=1}^{n} e_{i,s} \left(x_{i,s} + q_i\right) \Bigg) - f \left(\sum_{i=1}^{n} q_i - Q \right).
\end{aligned}
$$

To produce the building blocks of the KKT condition, we can use the partial derivatives of L with respect to the decision variables.

$$
\begin{aligned}
\frac{\partial L}{\partial q_i} &= -f - \sum_{s=1}^{S} e_{i,s} + (a_i + b q_i) + b \sum_{s=1}^{S} \theta_s x_{i,s} \\
\frac{\partial L}{\partial x_{i,s}} &= -e_{i,s} + \theta_s \left(-p_s + a_i + b q_i + (b + d_i) x_{i,s} \right) \\
\frac{\partial L}{\partial C_s} &= \left(p_s + Z C_s - Y_s \right) \theta_s \\
\frac{\partial L}{\partial Q} &= f - \sum_{s} \theta_s p_s \\
\frac{\partial L}{\partial p_s} &= \theta_s \left(C_s - \left(Q + \sum_{i=1}^{n} x_{i,s} \right) \right) \\
\frac{\partial L}{\partial f} &= Q - \sum_{i=1}^{n} q_i \\
\frac{\partial L}{\partial e_{i,s}} &= -q_i - x_{i,s}
\end{aligned}
$$

Thus, KKT of this problem can be represented as

$$
\begin{aligned}
& -f-\sum_{s=1}^{S} e_{i,s}+(a_i+bq_i)+b\sum_{s=1}^{S}\theta_s x_{i,s}=0 && \forall i\in\{1,\dots,n\} && \text{[C1]}\\
& Q=\sum_{i=1}^{n} q_i && && \text{[C2]}\\
& C_s=\left(Q+\sum_{i=1}^{n} x_{i,s}\right) && \forall s\in\{1,\dots,S\} && \text{[C3]}\\
& p_s=(Y_s-ZC_s) && \forall s\in\{1,\dots,S\} && \text{[C4]}\\
& f=\sum_{s=1}^{S}\theta_s p_s && && \text{[C5]}\\
& e_{i,s}=\theta_s\left(-p_s+a_i+bq_i+(b+d_i)x_{i,s}\right) && \forall i\in\{1,\dots,n\} && \text{[C6]}\\
& && \forall s\in\{1,\dots,S\} && \\
& e_{i,s}(q_i+x_{i,s})=0 && \forall i\in\{1,\dots,n\} && \text{[C7]}\\
& && \forall s\in\{1,\dots,S\} && \\
& e_{i,s}\geq 0 && \forall i\in\{1,\dots,n\} && \text{[C8]}\\
& && \forall s\in\{1,\dots,S\} && \\
& q_i+x_{i,s}\geq 0 && \forall i\in\{1,\dots,n\} && \text{[C9]}\\
& && \forall s\in\{1,\dots,S\}. &&
\end{aligned}
$$

If we replace the value of f and $e_{i,s}$ from [C5] and [C6] into [C1], constraint [C1] can be replaced with $\sum_{s=1}^{S}\theta_s x_{i,s}=0$.

5.1.2 Firms' optimisation problem

Problem WNN[j] represents the optimization problem solved by firm j to maximize its profit, subject to KKT conditions of ISO's optimization problem.

WNN[j]:

$$\max\ u_j = \sum_{s=1}^{S} \theta_s \Bigg(p_s(q_j + x_{j,s}) - \left(\alpha_j \left(q_j + x_{j,s}\right) + \frac{\beta_j}{2} \ \left(q_j + x_{j,s}\right)^2 + \frac{\delta_j}{2} {x_{j,s}}^2 \right) \Bigg)$$

$$\text{s.t.} \quad \sum_{s=1}^{S} \theta_s x_{i,s} = 0 \qquad \forall i \in \{1, \ldots, n\} \quad \text{[C1]}$$

$$Q = \sum_{i=1}^{n} q_i \qquad \text{[C2]}$$

$$C_s = \left(Q + \sum_{i=1}^{n} x_{i,s} \right) \qquad \forall s \in \{1, \ldots, S\} \quad \text{[C3]}$$

$$p_s = (Y_s - ZC_s) \qquad \forall s \in \{1, \ldots, S\} \quad \text{[C4]}$$

$$f = \sum_{s=1}^{S} \theta_s p_s \qquad \text{[C5]}$$

$$e_{i,s} = \theta_s \left(-p_s + a_i + bq_i + (b + d_i) x_{i,s}\right) \qquad \forall i \in \{1, \ldots, n\} \ \forall s \in \{1, \ldots, S\} \quad \text{[C6]}$$

$$e_{i,s}(q_i + x_{i,s}) = 0 \qquad \forall i \in \{1, \ldots, n\} \ \forall s \in \{1, \ldots, S\} \quad \text{[C7]}$$

$$e_{i,s} \geq 0 \qquad \forall i \in \{1, \ldots, n\} \ \forall s \in \{1, \ldots, S\} \quad \text{[C8]}$$

$$q_i + x_{i,s} \geq 0 \qquad \forall i \in \{1, \ldots, n\} \ \forall s \in \{1, \ldots, S\} \quad \text{[C9]}$$

To make the optimization problem look simpler, we can replace the values of Q, C_s, and f from [C2], [C3], and [C5] in the other equations. This simplifies

WNN to the following shape.

WNN[j]:

$$\max\ u_j = \sum_{s=1}^{S} \theta_s \Bigg(p_s(q_j + x_{j,s}) - \left(\alpha_j \left(q_j + x_{j,s}\right) + \frac{\beta_j}{2}\ \left(q_j + x_{j,s}\right)^2 + \frac{\delta_j}{2} {x_{j,s}}^2 \right) \Bigg)$$

$$\begin{aligned}
\text{s.t.}\quad & \sum_{s=1}^{S} \theta_s x_{i,s} = 0 && \forall i \in \{1,\dots,n\} && \text{[C1]}\\
& p_s = Y_s - Z\left(\sum_{h=1}^{n} q_h + \sum_{h=1}^{n} x_{h,s}\right) && \forall s \in \{1,\dots,S\} && \text{[C4]}\\
& e_{i,s} = -\theta_s\left(-p_s + a_i + bq_i + (b+d_i)x_{i,s}\right) && \forall i \in \{1,\dots,n\} && \text{[C6]}\\
& && \forall s \in \{1,\dots,S\} && \\
& e_{i,s}(q_i + x_{i,s}) = 0 && \forall i \in \{1,\dots,n\} && \text{[C7]}\\
& && \forall s \in \{1,\dots,S\} && \\
& e_{i,s} \geq 0 && \forall i \in \{1,\dots,n\} && \text{[C8]}\\
& && \forall s \in \{1,\dots,S\} && \\
& q_i + x_{i,s} \geq 0 && \forall i \in \{1,\dots,n\} && \text{[C9]}\\
& && \forall s \in \{1,\dots,S\} &&
\end{aligned}$$

With a similar process, the optimization problem of firm j in a stochastic market clearing mechanism without non-negativity constraints can be found as

WONN[j]:

$$\max\ u_j = \sum_{s=1}^{S} \theta_s \Bigg(p_s(q_j + x_{j,s}) - \left(\alpha_j \left(q_j + x_{j,s}\right) + \frac{\beta_j}{2}\ \left(q_j + x_{j,s}\right)^2 + \frac{\delta_j}{2} {x_{j,s}}^2 \right) \Bigg)$$

$$\begin{aligned}
\text{s.t.}\quad & \sum_{s=1}^{S} \theta_s x_{i,s} = 0 && \forall i \in \{1,\dots,n\} && \text{[C1]}\\
& p_s = Y_s - Z\left(\sum_{h=1}^{n} q_h + \sum_{h=1}^{n} x_{h,s}\right) && \forall s \in \{1,\dots,S\} && \text{[C4]}\\
& e_{i,s} = \theta_s\left(-p_s + a_i + bq_i + (b+d_i)x_{i,s}\right) && \forall i \in \{1,\dots,n\} && \text{[C6]}\\
& e_{i,s} = 0 && \forall i \in \{1,\dots,n\} && \text{[C11]}\\
& && \forall s \in \{1,\dots,S\} &&
\end{aligned}$$

Also, we introduce a relaxation to WNN, which we use later in proofs of our theorems. We eliminate constraint [C7]: $e_{i,s}(q_i + x_{i,s}) = 0$, and limit the constraint [C9]: $\forall i, q_i + x_{i,s} \geq 0$ to the optimizer generator j to obtain a relaxation problem as follows.

RWNN:

$$\begin{aligned}
\max\ u_j = \sum_{s=1}^{S} \theta_s \Bigg(& p_s(q_j + x_{j,s}) - \\
& \left(\alpha_j \left(q_j + x_{j,s}\right) + \frac{\beta_j}{2}\ \left(q_j + x_{j,s}\right)^2 + \frac{\delta_j}{2} {x_{j,s}}^2 \right) \Bigg)
\end{aligned}$$

$$\begin{aligned}
\text{s.t.}\quad & \sum_{s=1}^{S} \theta_s x_{i,s} = 0 && \forall i \in \{1, \dots, n\} && \text{[C1]} \\
& p_s = Y_s - Z \left(\sum_{h=1}^{n} q_h + \sum_{h=1}^{n} x_{h,s} \right) && \forall s \in \{1, \dots, S\} && \text{[C4]} \\
& f = \sum_{s=1}^{S} \theta_s p_s && && \text{[C5]} \\
& e_{i,s} = \theta_s \left(-p_s + a_i + b q_i + (b + d_i) x_{i,s} \right) && \forall i \in \{1, \dots, n\} && \text{[C6]} \\
& && \forall s \in \{1, \dots, S\} && \\
& e_{i,s} \geq 0 && \forall i \in \{1, \dots, n\} && \text{[C8]} \\
& q_j + x_{j,s} \geq 0 && \forall s \in \{1, \dots, S\} && \text{[C12]}
\end{aligned}$$

Now, we prove three lemmas which help us to demonstrate the final theorem.

Lemma 5.1. *If for every $i \neq j$(j is the optimizer generator), a_i and d_i has the same value, then the constraint $e_{i,s} \geq 0$ (for every $i \neq j$) in RWNN can be replaced with $e_{i,s} = 0$ without reducing the optimal value of RWNN.*

Proof. We prove the lemma by contradiction. Assume there exist a point $\nu = (a_j, d_j, \boldsymbol{q}, \boldsymbol{x}, \boldsymbol{p}, \boldsymbol{e})$ with at least one $e_{i',s'} > 0$ ($i \neq j$) and higher objective value than any feasible solution with $\boldsymbol{e} = \boldsymbol{0}$.

Consider $\nu' = (a'_j, d'_j, \boldsymbol{q}', \boldsymbol{x}', \boldsymbol{p}', \boldsymbol{e}')$ defined as follows.

$$q'_i = \begin{cases} q_i & i = j \\ q_i + \frac{\frac{Z\sum_{h\neq j}\sum_w e_{h,w}}{Z(n-1)+b} - \sum_w e_{i,w}}{b} & i \neq j \end{cases} \tag{36}$$

$$x'_{i,s} = \begin{cases} x_{i,s} & i = j \\ x_{i,s} + \frac{\sum_w e_{i,w} - \frac{e_{i,s}}{\theta_s}}{b+d_i} - \frac{Z(\sum_{h\neq j}\sum_w e_{h,w} - \sum_{h\neq j}\frac{e_{h,s}}{\theta_s})}{(Z(n-1)+b+d_i)(b+d_i)} & i \neq j \end{cases} \tag{37}$$

$$\begin{aligned} a'_j \geq \max_s \ & \Bigg\{ Z\Bigg(\sum_{h\neq j}\sum_w e_{h,w}\left(\frac{1}{Z(n-1)+b} - \frac{1}{Z(n-1)+b+d_i}\right) \\ & + \frac{\sum_{h\neq j}\frac{e_{h,s}}{\theta_s}}{Z(n-1)+b+d_i}\Bigg) + a_j\Bigg\} \end{aligned} \tag{38}$$

$$d'_j = d_j \tag{39}$$

Firstly, we show this is a feasible solution.

$$\sum_s \theta_s x'_{i,s} = \begin{cases} \sum_s \theta_s x_{i,s} & i = j \\ \sum_s \theta_s x_{i,s} + \frac{\sum_w e_{i,w} - \sum_s \theta_s \frac{e_{i,s}}{\theta_s}}{b+d_i} - \frac{Z(\sum_{h\neq j}\sum_w e_{h,w} - \sum_s \theta_s \sum_{h\neq j}\frac{e_{h,s}}{\theta_s})}{(Z(n-1)+b+d_i)(b+d_i)} & i \neq j \end{cases}$$

Extra simplifications yield

$$\sum_s \theta_s x'_{i,s} = 0 \qquad \forall i. \tag{40}$$

After substituting the value of q'_h from (36) into $\sum_{h\neq j} q'_h$ and slightly simplifying the resulted equation, we get

$$\sum_{h\neq j} q'_h = \sum_{h\neq j} q_h - \frac{\sum_{h\neq j}\sum_w e_{h,w}}{Z(n-1)+b}. \tag{41}$$

The same analysis on equation (37) gives us the following equation:

$$\sum_{h\neq j} x'_{h,s} = \sum_{h\neq j} x_{h,s} + \frac{\sum_{h\neq j}\sum_w e_{h,w} - \sum_{h\neq j}\frac{e_{h,s}}{\theta_s}}{Z(n-1)+b+d_i}. \tag{42}$$

The quantity p'_s can be obtained combining equations [C4], (41), and (42).

$$\begin{aligned} p'_s =& p_s - Z\left(-\frac{\sum_{h\neq j}\sum_w e_{h,w}}{Z(n-1)+b} + \frac{\sum_{h\neq j}\sum_w e_{h,w} - \sum_{h\neq j}\frac{e_{h,s}}{\theta_s}}{Z(n-1)+b+d_i}\right) \\ =& p_s + Z\Bigg(\sum_{h\neq j}\sum_w e_{h,w}\left(\frac{1}{Z(n-1)+b} - \frac{1}{Z(n-1)+b+d_i}\right) \\ & + \frac{\sum_{h\neq j}\frac{e_{h,s}}{\theta_s}}{Z(n-1)+b+d_i}\Bigg) \end{aligned} \tag{43}$$

Considering the fact that $e_{i,s}$, Z, b, and d_i have non-negative values,

$$p'_s \geq p_s. \tag{44}$$

From (36), (37), (43), and [C6], $e_{i,s}$ can be obtained as follows.

$$e'_{i,s} = \begin{cases} e_{i,s} + \theta_s(-p'_s + p_s + a'_j - a_j) & i = j \\ e_{i,s} + \theta_s\Bigg(- Z\sum_{h\neq j}\sum_w e_{h,w}\left(\frac{1}{Z(n-1)+b} - \frac{1}{Z(n-1)+b+d_i}\right) & i \neq j \\ \quad -Z\frac{\sum_{h\neq j}\frac{e_{h,s}}{\theta_s}}{Z(n-1)+b+d_i} + \frac{Z\sum_{h\neq j}\sum_w e_{h,w}}{Z(n-1)+b} - \sum_w e_{i,w} & \\ \quad +\sum_w e_{i,w} - \frac{e_{i,s}}{\theta_s} - \frac{Z(\sum_{h\neq j}\sum_w e_{h,w} - \sum_{h\neq j}\frac{e_{h,s}}{\theta_s})}{Z(n-1)+b+d_i}\Bigg) & \end{cases}$$

This simplifies to

$$e'_{i,s} = \begin{cases} e'_{j,s} \geq 0 & i = j \\ 0 & i \neq j. \end{cases}$$

Thus, the constraint [C8] is also satisfied. As $q'_j = q_j$, $x'_{j,s} = x_{j,s}$, and ν is a feasible solution, constraints [C12] are also fulfilled. In sum, ν' is a feasible solution.

On the other hand, a comparison between the u'_j and u_j demonstrates that ν' gives a better objective:

$$u'_j - u_j = \sum_s \theta_s(p'_s - p_s)(q_j + x_{j,s}).$$

With $q_j + x_{j,s} \geq 0$, as concluded from [C12], and $p'_s - p_s \geq 0$ as resolved in (44)

$$u'_j \geq u_j$$

This contradicts the initial assumption, which proves the lemma. □

Lemma 5.2. *RWNN can be simplified to the following optimization problem.*

RWNN:

$$
\begin{aligned}
\max\ u_j = &f q_j + \sum_{s=1}^{S} \theta_s (p_s - f) x_{j,s} \\
&- \left(\alpha_j q_j + \frac{\beta_j}{2} q_j^2 + \frac{\beta_j + \delta_j}{2} \sum_{s=1}^{S} \theta_s {x_{j,s}}^2 \right)
\end{aligned}
$$

$$
\begin{aligned}
\textit{s.t.}\quad & \sum_{s=1}^{S} \theta_s x_{i,s} = 0 && \forall i \in \{1, \dots, n\} && \textit{[C1]} \\
& p_s = Y_s - Z \left(\sum_{h=1}^{n} q_h + \sum_{h=1}^{n} x_{h,s} \right) && \forall s \in \{1, \dots, S\} && \textit{[C4]} \\
& f = \sum_{s=1}^{S} \theta_s p_s && && \textit{[C5]} \\
& e_{i,s} = \theta_s \left(-p_s + a_i + b q_i + (b + d_i) x_{i,s} \right) && \forall i \in \{1, \dots, n\} && \textit{[C6]} \\
& && \forall s \in \{1, \dots, S\} && \\
& e_{i,s} \geq 0 && \forall i \in \{1, \dots, n\} && \textit{[C8]} \\
& q_j + x_{j,s} \geq 0 && \forall s \in \{1, \dots, S\} && \textit{[C12]}
\end{aligned}
$$

Proof. The first part of the objective function is the optimizer's income, which is equal to

$$
\begin{aligned}
\sum_{s=1}^{S} \theta_s p_s (q_j + x_{j,s}) = & \sum_{s=1}^{S} \theta_s p_s q_j + \sum_{s=1}^{S} \theta_s p_s x_{j,s} \\
= & f q_j + \sum_{s=1}^{S} \theta_s f x_{j,s} + \sum_{s=1}^{S} \theta_s (p_s - f) x_{j,s} && \text{From [C5]} \\
= & f q_j + \sum_{s=1}^{S} \theta_s (p_s - f) x_{j,s} + f \sum_{s=1}^{S} \theta_s x_{j,s} \\
= & f q_j + \sum_{s=1}^{S} \theta_s (p_s - f) x_{j,s}. && \text{From [C1]}
\end{aligned}
$$

The rest of the objective function can also be simplified similarly, as follows.

$$\begin{aligned}\text{Generating Cost} =& \sum_{s=1}^{S} \theta_s \left(\alpha_j \left(q_j + x_{j,s}\right) + \frac{\beta_j}{2} \ \left(q_j + x_{j,s}\right)^2 + \frac{\delta_j}{2} {x_{j,s}}^2 \right) \\ =& \alpha_j q_j + \frac{\beta_j}{2} q_j^2 + \frac{\beta_j + \delta_j}{2} \sum_{s=1}^{S} \theta_s {x_{j,s}}^2 \\ &+ \left(\alpha_j + \beta_j q_j\right) \sum_{s=1}^{S} \theta_s x_{j,s} \\ =& \alpha_j q_j + \frac{\beta_j}{2} q_j^2 + \frac{\beta_j + \delta_j}{2} \sum_{s=1}^{S} \theta_s {x_{j,s}}^2. && \text{From [C1]}\end{aligned}$$

□

Lemma 5.3. *If for every $i \neq j$ (j is the optimizer generator), a_i and d_i has the same value, then the optimal solution to WONN is at least as good as the optimal value to RWNN.*

Proof. To prove the lemma, we find the optimal solution to RWNN, while we ignore the non-negativity constraint $q_j + x_{j,s} \geq 0$. Thus, this point gives an objective value as good as (possibly better than) the optimal point. Then we show this point is a feasible solution for WONN, which proves the lemma.

From Lemma 5.2 we have

$$e_{i,s} = \theta_s \left(-Y_s + Z \left(\sum_{h=1}^{n} q_h + \sum_{h=1}^{n} x_{h,s} \right) + a_i + b q_i + (b + d_i) x_{i,s} \right).$$

To simplify the equations we use some transformations. Let $R_i = \frac{1}{(b+d_i)}$, and $A_i = \frac{a_i}{b}$. Also, let A and R denote $\sum_{h=1}^{n} A_h$, and $\sum_{h=1}^{n} R_h$ respectively. Then, constraint [C6] looks like

$$e_{i,s} = \theta_s \left(-Y_s + Z \left(\sum_{h=1}^{n} q_h + \sum_{h=1}^{n} x_{h,s} \right) + \frac{1}{R_i} x_{i,s} + b \left(A_i + q_i\right) \right). \tag{45}$$

A summation over different scenarios gives

$$\sum_{w=1}^{S} e_{i,w} = -Y + Z \sum_{h=1}^{n} q_i + \left(A_i + q_i\right) b. \tag{46}$$

From Lemma 5.1, the constraints $e_{i,s} = 0$ for every $i \neq j$ and s can be replaced with $e_{i,s} \geq 0$ in RWNN. On the other hand, from the assumption we know that A_i has a fixed value for every $i \neq j$. As a result, equation (46) is used to show that q_i must have a fixed value for every $i \neq j$. Thus, equation (46) can be re-written as

$$0 = -Y + Z\left((n-1)q_i + q_j\right) + \left(A_i + q_i\right)b. \tag{47}$$

With a similar argument, we can show that $x_{i,s}$ also has the same value for every $i \neq j$. Equation (45), thus, can be represented as

$$0 = \theta_s \left(-Y_s + Z\left((n-1)q_i + q_j + (n-1)x_{i,s} + x_{j,s}\right) + \frac{1}{R_i}x_{i,s} + b\left(A_i + q_i\right)\right) \tag{48}$$

Solving equations (47) and (48), we find the values of q_i and $x_{i,s}$ as functions of q_j and $x_{j,s}$.

$$\begin{aligned} q_i =& \frac{Y - bA_i - Zq_j}{b + (n-1)Z} \\ x_{i,s} =& -\frac{R_i\left(Y - Y_s + Zx_{j,s}\right)}{1 + (n-1)ZR_i} \end{aligned} \tag{49}$$

From (49) we can also calculate the values of f and $p_s - f$ as functions of q_j and $x_{j,s}$.

$$\begin{aligned} f =& \frac{b\left(Y + (n-1)ZA_i - Zq_j\right)}{b + (n-1)Z} \\ p_s - f =& \frac{-Y + Y_s - Zx_{j,s}}{1 + (n-1)ZR_i} \end{aligned} \tag{50}$$

Inserting these values into the utility function from Lemma 5.2 simplifies the utility function to

$$\begin{aligned} u_j = &\left(\frac{b\left(Y + (n-1)ZA_i - Zq_j\right)}{b + (n-1)Z} - \alpha_j - \frac{\beta_j}{2}q_j\right)q_j \\ &+ \sum_{s=1}^{S}\theta_s\left(\frac{-Y + Y_s - Zx_{j,s}}{1 + (n-1)ZR_i} - \frac{\beta_j + \delta_j}{2}x_{j,s}\right)x_{j,s} \end{aligned}$$

As Z, α_j, β_j, and R_i have non-negative values, u_j is a concave function of q_j and $\boldsymbol{x}_j$. Therefore, ignoring the rest of the constraints, the optimal value of q_j and $x_{j,s}$ can be found using first order conditions.

First order conditions for q_j and $x_{j,s}$ gives

$$q_j^* = \frac{bY + (n-1)bZA_i - (b + (n-1)Z)\alpha_j}{2bZ + (b + (n-1)Z)\beta_j} \tag{51}$$

$$x_{j,s}^* = \frac{Y_s - Y}{2Z + \left(1 + (n-1)ZR_i\right)\left(\beta_j + \delta_j\right)}. \tag{52}$$

Now we need to show that we can always find A_j and R_j, so that this value is a feasible solution to WONN and yields $e_{j,s} = 0$. To do so, we first calculate

$\frac{e_{j,s}}{\theta_s} - \sum_w e_{j,w}$ for all s. From (45), (46), and (51)

$$\begin{aligned}\frac{e_{j,s}}{\theta_s} - \sum_{w=1}^{S} e_{j,w} =& Y - Y_s + \frac{x_{j,s}}{R_j} + Z\left((n-1)x_{i,s} + x_{j,s}\right) \\ =& \frac{(Y-Y_s)\left(-1 + R_j\left(Z + \beta_j + \delta_j\right) + (n-1)ZR_i\left(-1 + R_j\left(\beta_j + \delta_j\right)\right)\right)}{R_j\left(1 + (n-1)ZR_i\right)\left(2Z + \left(1 + (n-1)ZR_i\right)\left(\beta_j + \delta_j\right)\right)}.\end{aligned} \tag{53}$$

It is always possible to choose R_j as follows to ensure that $\frac{e_{j,s}}{\theta_s} - \sum_w e_{j,w} = 0$. Note that this does not change either of production quantities or prices. This value of R_j is

$$R_j = \frac{1 + (n-1)ZR_i}{Z + \left(1 + (n-1)ZR_i\right)\left(\beta_j + \delta_j\right)}.$$

We can also choose A_j so that $\sum_w e_{j,w} = 0$ without changing any production quantity and thus any prices. From (46) and (49)

$$\begin{aligned}\sum_{w=1}^{S} e_{j,w} =& -Y + Z\left((n-1)q_i + q_j\right) + b\left(A_h + q_h\right) \\ =& -Y + b\left(A_j + q_j\right) + \frac{(n-1)Z\left(Y - bA_i\right) + bZq_j}{b + (n-1)Z}.\end{aligned} \tag{54}$$

Solving $\sum_w e_{j,w} = 0$ for A_j gives

$$\begin{aligned}A_j =& \frac{-bY(b + (n-2)Z) + (b + (n-1)Z)\left((b + nZ)\alpha_j + Y\beta_j\right)}{(b + (n-1)Z)\left(2bZ + (b + (n-1)Z)\beta_j\right)} \\ &+ \frac{-(n-1)ZA_i\left(b(b + (n-2)Z) - (b + (n-1)Z)\beta_j\right)}{(b + (n-1)Z)\left(2bZ + (b + (n-1)Z)\beta_j\right)}.\end{aligned}$$

These values of A_j and R_j ensure that $e_{j,s} = 0\,\forall s$. Thus, constraints [C6] and [C8] are met in WONN and RWNN.

From (51) we derive $\sum_s \theta_s x_{j,s} = 0$. We can use the fact that $\sum_s \theta_s x_{j,s} = 0$ to show that for $i \neq j$ also $\sum_s \theta_s x_{i,s} = 0$ (in equation (49)). So this optimal point is feasible in [C1]. In sum, the constructed point is feasible for WONN, and gives an objective value at least as good as RWNN. □

Now, we can use the above lemmas to prove a theorem that justifies using the equilibrium of the simplified game without the non-negativity constraints instead of the equilibrium of the original game.

Theorem. *The equilibrium of the symmetric stochastic settlement game without the non- negativity constraints in ISOSP-FD is also the equilibrium of the stochastic settlement game with the non-negativity constraints.*

Proof. To prove the theorem, we should show that if all generators offer the equilibrium values of a_i and d_i, none of them are willing to deviate from it. Equivalently, if in WNN a_i and d_i are equal to the equilibrium of the SFSP game without the non-negativity constraints for all $i \neq j$, then optimal a_j and d_j are also equal to equilibrium values of this game.

The equilibrium of SFSP without non-negativity constraints is equal to the optimal value of WONN when every non-optimizer generator has offered the equilibrium values of the game. Thus, we prove that the optimal value of WONN is also optimal to WNN.

Firstly, Lemma 5.3 states that if the optimal solution to WONN is feasible to RWNN, then, it is also the optimal solution to RWNN. In our problem, from Proposition 8 we know that the optimal solution to WONN holds both $q_i \geq 0$ and $q_i + x_{i,s} \geq 0$. The other constraints of RWNN are shared between these two models. Thus, it is feasible and optimal in RWNN.

On the other hand, every feasible solution to WNN is feasible in RWNN. So, if this solution (which is the optimal solution to RWNN) is feasible to WNN, then it is also optimal to WNN. From Proposition 8, we know that $q_i \geq 0$ and $q_i + x_{i,s} \geq 0$ for all i, as it is the equilibrium of the game without non-negativity constraints. This means this point is feasible in [C8] and [C9]. On the other hand, we know that $e_{i,s} = 0$ for all i, as it is the optimal solution to WONN. This shows it also holds [C7]. The other constraints are common and thus met. In sum, This point is feasible and therefore optimal to WNN.

Thus, no generator is willing to deviate from this point unilaterally, and this is the equilibrium of WNN. □